The Differential of a Function of Several Variables

Guoning Wu

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1 作業

1. 求下列函數的偏導數
   (a) $z = x^2 y$
   (b) $z = y \cos x$
   (c) $z = \ln(x + y^2)$
   (d) $z = \arctan \dfrac{y}{x}$
   (e) $u = (xy)^z$
   (f) $u = x^{y^z}$
2. 設
$$f(x,y) = \begin{cases} y \sin \frac{1}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0 \end{cases}$$
   考察函數在$(0,0)$點的偏導數。
3. 證明函數$z = \sqrt{x^2+y^2}$在$(0,0)$點連續但偏導數不存在。
4. 考察函數
$$f(x,y) = \begin{cases} xy \sin \frac{1}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0 \end{cases}$$
   在$(0,0)$點的可微性。

5. 證明函數

$$f(x,y)=\begin{cases}\frac{x^2y}{x^2+y^2}, & x^2+y^2\neq 0\\ \\ 0, & x^2+y^2=0\end{cases}$$

在$(0,0)$點的連續，偏導數存在，但在該點不可微。

6. 驗證函數

$$f(x,y)=\begin{cases}\frac{xy}{\sqrt{x^2+y^2}}, & x^2+y^2\neq 0\\ \\ 0, & x^2+y^2=0\end{cases}$$

在$(0,0)$點連續且可偏導，但在該點不可微。

7. 驗證函數

$$f(x,y)=\begin{cases}(x^2+y^2)\sin\frac{1}{x^2+y^2}, & x^2+y^2\neq 0\\ \\ 0, & x^2+y^2=0\end{cases}$$

的偏導函數$f_x(x,y), f_y(x,y)$在$(0,0)$點不連續，但在該點可微。

8. 驗證函數

$$f(x,y)=\begin{cases}\frac{2xy^2}{x^2+y^4}, & x^2+y^2\neq 0\\ \\ 0, & x^2+y^2=0\end{cases}$$

在$(0,0)$點沿著各個方向的方向導數存在，但它在該點不連續，故在該點不可微。

9. 計算下列函數的梯度
   (a) $z=x^2+y^2\sin(xy)$
   (b) $u=x^2+2y^2+3xy+4yz+6x-2y-5z$在點$(1,1,1)$

10. 計算下列函數的高階導數
    (a) $z=\arctan\dfrac{y}{x}$，求$\dfrac{\partial^2 z}{\partial x^2},\dfrac{\partial^2 z}{\partial x\partial y},\dfrac{\partial^2 z}{\partial y^2}$
    (b) $z=xe^{xy}$，求$\dfrac{\partial^2 z}{\partial x^2},\dfrac{\partial^2 z}{\partial x\partial y},\dfrac{\partial^2 z}{\partial y^2}$
    (c) $u=xyze^{x+y+z}$，求$\dfrac{\partial^{p+q+r}u}{\partial x^p\partial y^p\partial z^r}$

11. 計算下列函數的高階微分

   (a) $z = x\ln(xy)$, 求$\mathrm{d}^2 z$

   (b) $z = \sin^2(ax+by)$, 求$\mathrm{d}^3 z$

12. 利用鏈式法則求下列函數的極限

   (a) $z = u^2\ln v, u = \dfrac{x}{y}, v = 3x-2y$，求$\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$

   (b) $z = x^2+y^2+\cos(x+y), x = u+v, y = \arcsin v$，求$\dfrac{\partial z}{\partial u}, \dfrac{\partial^2 z}{\partial u \partial v}$

   (c) $u = f(xy, \dfrac{x}{y})$，求$\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial^2 u}{\partial x \partial y}, \dfrac{\partial^2 u}{\partial y^2}$

13. $z = f(x,y)$具有連續的二階偏導數，出$\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2}$ 在座標變換

$$\begin{cases} u = x^2 - y^2 \\ u = 2xy \end{cases}$$

   下的表達式。

14. 設向量值函數$f : \mathbb{R}^2 \to \mathbb{R}^3$的座標分量函數為：

$$\begin{cases} x = u^2 + v^2 \\ y = u^2 - v^2 \\ z = uv \end{cases}$$

   向量值函數$g : \mathbb{R}^2 \to \mathbb{R}^2$的座標分量表示為：

$$\begin{cases} u = r\cos\theta \\ v = r\sin\theta \end{cases}$$

   求復合函數$f \circ g$的導數。

15. 設$u = f(x,y), x = r\cos\theta, y = r\sin\theta$，證明：

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

16. 求函數$f(x,y) = 2x^2 - xy - y^2 - 6x - 3y + 5$在點$(1,-2)$的泰勒公式

17. 求下列函數的極值點

(a) $z = 3axy - x^3 - y^3 (a > 0)$

(b) $z = x^2 - xy - 2x + y (a > 0)$

18. 求下列方程所確定的隱函數的導數或偏導數

(a) $\sin y + e^x - xy^2 = 0$，求$\frac{\mathrm{d}y}{\mathrm{d}x}$

(b) $x^y = y^x$，求$\frac{\mathrm{d}y}{\mathrm{d}x}$

(c) $e^z - xyz = 0$，求$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x \partial y}$

(d) $f(x, x+y, x+y+z) = 0$，求$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x \partial y}$

19. 求下列方程組所確定的隱函數的導數或偏導數

(a) $\begin{cases} z - x^2 - y^2 = 0 \\ x^2 + 2y^2 + 3z^2 = 4a^2 \end{cases}$,求$\dfrac{\mathrm{d}y}{\mathrm{d}x}, \dfrac{\mathrm{d}z}{\mathrm{d}x}, \dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}, \dfrac{\mathrm{d}^2 z}{\mathrm{d}x^2}$

(b) $\begin{cases} xu + yv = 0 \\ yu + xv = 1 \end{cases}$,求$\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial^2 u}{\partial x \partial y}, \dfrac{\partial^2 u}{\partial x^2}$

(c) $\begin{cases} u = f(ux, v+y) \\ v = g(u-x, v^2 y) \end{cases}$,求$\dfrac{\partial u}{\partial x}, \dfrac{\partial v}{\partial x}$

20. 通過變量替換$\begin{cases} x = e^{\xi} \\ y = e^{\eta} \end{cases}$變換方程

$$ax^2 \frac{\partial^2 z}{\partial x^2} + 2bxy \frac{\partial^2 z}{\partial x \partial y} + cy^2 \frac{\partial^2 z}{\partial y^2} = 0. (a, b, c \in \mathbb{R})$$

21. 求下列曲線在所示點處的切線與發平面

(a) $x = a \sin^2 t, y = b \sin t \cos t, z = c \cos^2 t$在點$t = \dfrac{\pi}{4}$.

(b) $2x^2 + 3y^2 + z^2 = 9, z^2 = 3x^2 + y^2$在點$(1, -1, 2)$

22. 求曲面$x^2 + 2y^2 + 3z^2 = 21$的切平面，使它平行與平面$x + 4y + 6z = 0$

23. 求$f(x, y, z) = xyz$在條件$\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = \dfrac{1}{r} (x > 0, y > 0, z > 0, r > 0)$下的極小值，並證明不等式

$$3 \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right)^{-1} \leq \sqrt[3]{abc},$$

其中a, b, c為任意的正實數。

24. 應用拉格朗日乘數法，求下列函數的條件極值

(a) $f(x, y) = x^2 + y^2, s.t.$(subject to)$x + y - 1 = 0$

(b) $f(x, y) = xyz, s.t.$(subject to)$x^2 + y^2 + z^2 = 1, x + y + z = 0$