Improper Integrals

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1 What is Improper Integrals

Definition 1.1. Suppose that $x \to f(x)$ is defined on the interval $[a, +\infty)$ and integrable on every closed interval $[a, b]$ contained in that interval. If the limit below exists,

$$\int_a^{+\infty} f(x)\,\mathrm{d}x = \lim_{b \to +\infty} \int_a^b f(x)\,\mathrm{d}x,$$

we call it the improper Riemann integral or the improper integral of the function f over the interval $[a, +\infty)$.

The expression $\int_a^\infty f(x)\,\mathrm{d}x$ itself is also called an improper integral, and in that case:

1. The integral converges if the limit exists;
2. The integral diverges if the limit does not exist.

Examples 1. Let us investigate the values of the parameter α for which the improper integral

$$\int_1^{+\infty} \frac{1}{x^\alpha}\,\mathrm{d}x$$

converges.

Definition 1.2. Suppose that $x \to f(x)$ is defined on the interval $[a, B)$ and integrable on any closed interval $[a, b] \subset [a, B)$. If the limit below exists:

$$\int_a^B f(x)\,\mathrm{d}x = \lim_{b \to B-0} \int_a^b f(x)\,\mathrm{d}x,$$

we call it the improper integral of f over the interval $[a, B)$.

Similarly, if a function $x \to f(x)$ is defined on the interval $(A, b]$ and integrable on every closed interval $[a, b] \subset (A, b]$, then we can define:

$$\int_A^b f(x)\,\mathrm{d}x = \lim_{a \to A+0} \int_a^b f(x)\,\mathrm{d}x.$$

and also by definition we set

$$\int_{-\infty}^b f(x)\,\mathrm{d}x = \lim_{a \to -\infty} \int_a^b f(x)\,\mathrm{d}x$$

Examples 2. Let us investigate the values of the parameters α for which the integral

$$\int_0^1 \frac{1}{x^\alpha}\,\mathrm{d}x$$

converges.

Examples 3. Let us investigate the values of the parameters α for which the integral

$$\int_0^{+\infty} e^{-\alpha x}\,\mathrm{d}x$$

converges.

Examples 4. Compute the integral

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,\mathrm{d}x.$$

Examples 5. Let us investigate the integral

$$\int_{-\infty}^{+\infty} \frac{e^{\frac{1}{x}}}{x^2}\,\mathrm{d}x.$$

Examples 6. Compute the integral

$$\int_0^{+\infty} e^{-x} x^n\,\mathrm{d}x.(n \in \mathbb{Z}^+)$$

Examples 7. Compute the integral

$$\int_0^1 \ln x\,\mathrm{d}x.$$

Examples 8. Compute the integral

$$\int_0^{\frac{\pi}{2}} \ln \sin x\,\mathrm{d}x.$$

Examples 9. Compute the integral

$$\int_0^{+\infty} \frac{1}{(1+x^2)(1+x^\alpha)}\,\mathrm{d}x.$$

2 Convergence of an Improper Integral

2.1 The Cauchy Criterion

Let $[a,\omega)$ be a finite or infinite interval and $x \to f(x)$ a function defined on that interval and integrable over every closed interval $[a,b] \subset [a,\omega)$. Then by definition

$$\int_a^\omega f(x)\,\mathrm{d}x = \lim_{b\to\omega} \int_a^b f(x)\,\mathrm{d}x, \tag{1}$$

if this limit exists as $b \to \omega, b \in [a,\omega)$.

The convergence of the improper integral $\int_a^\omega f(x)\,\mathrm{d}x$ is equivalent to the existence of a limit for the function

$$\mathcal{F}(b) = \int_a^b f(x)\,\mathrm{d}x \tag{2}$$

as $b \to \omega, b \in [a,\omega)$.

Proposition 2.1. If the function $x \to f(x)$ is defined on the interval $[a,\omega)$ and integrable on every closed interval $[a,b] \subset [a,\omega)$, then the integral $\int_a^\omega f(x)\,\mathrm{d}x$ converges if and only is for every $\epsilon > 0$ there exists $B \in [a,\omega)$, such that the relation

$$\left|\int_{b_1}^{b_2} f(x)\,\mathrm{d}x\right| \leq \epsilon$$

for any $b_1, b_2 \in [a,\omega)$ satisfying $B < b_1$ and $B < b_2$.

2.2 Absolute Convergence of an Improper Integral

Definition 2.1. The improper integral $\int_a^\omega f(x)\,\mathrm{d}x$ converges absolutely if the integral $\int_a^\omega |f|\,\mathrm{d}x$ converges.

Proposition 2.2. If a function $f \geq 0$ and integrable on every $[a,b] \subset [a,\omega)$, then the improper integral $\int_a^\omega f(x)\,\mathrm{d}x$ exists if and only if the function $\mathcal{F}(b) = \int_a^b f(x)\,\mathrm{d}x$ is bounded on $[a,\omega)$.

Theorem 2.1. Suppose that the function $x \to f(x)$ and $x \to g(x)$ are defined on the interval $[a,\omega)$ and integrable on any closed interval $[a,b] \subset [a,\omega)$. If

$$0 \leq f(x) \leq g(x)$$

on $[a,\omega)$, then the convergence of $\int_a^\omega g(x)\,\mathrm{d}x$ implies convergence of $\int_a^\omega f(x)\,\mathrm{d}x$, and the inequality

$$\int_a^\omega f(x)\,\mathrm{d}x \leq \int_a^\omega g(x)\,\mathrm{d}x$$

holds. Divergence of the integral $\int_a^\omega f(x)\,\mathrm{dx}$ implies divergence of $\int_a^\omega g(x)\,\mathrm{d}x$.

Examples 10. Let us discuss the integral

$$\int_0^{+\infty} \frac{\sqrt{x}}{\sqrt{1+x^4}}\,\mathrm{d}x$$

Examples 11. Let us discuss the integral

$$\int_1^{+\infty} \frac{\cos x}{x^2}\,\mathrm{d}x$$

Examples 12. Let us discuss the integral

$$\int_1^{+\infty} e^{-x^2}\,\mathrm{d}x$$

Examples 13. Let us discuss the integral

$$\int_e^{+\infty} \frac{1}{\ln x}\,\mathrm{d}x$$

Examples 14. Let us discuss the Euler integral

$$\int_0^{\frac{\pi}{2}} \ln \sin x\,\mathrm{d}x$$

Examples 15. Let us discuss the elliptic integral

$$\int_0^1 \frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}}\,\mathrm{d}x(0<k^2<1)$$

Examples 16. Let us discuss the integral

$$\int_0^1 \frac{1}{\cos\theta - \cos\varphi}\,\mathrm{d}x$$

3 Conditional Convergence of an Improper Integral

Definition 3.1. If an improper integral converges but not absolutely, we say that it converges conditionally.

Examples 17. The integral

$$\int_{\frac{\pi}{2}}^{+\infty} \frac{\sin x}{x}\,\mathrm{d}x = -\left.\frac{\cos x}{x}\right|_{\frac{\pi}{2}}^{+\infty} - \int_{\frac{\pi}{2}}^{+\infty} \frac{\cos x}{x^2}\,\mathrm{d}x = -\int_{\frac{\pi}{2}}^{+\infty} \frac{\cos x}{x^2}\,\mathrm{d}x$$

The integral above converges, however,

$$\int_{\frac{\pi}{2}}^{+\infty}\left|\frac{\sin x}{x}\right| \mathrm{d}x \geq \int_{\frac{\pi}{2}}^{+\infty}\frac{\sin^2 x}{x}\,\mathrm{d}x = \frac{1}{2}\int_{\frac{\pi}{2}}^{+\infty}\frac{1}{x}\,\mathrm{d}x - \frac{1}{2}\int_{\frac{\pi}{2}}^{+\infty}\frac{\cos 2x}{x}\,\mathrm{d}x$$

The integral is not absolutely convergent.

Proposition 3.1. Let $x \to f(x)$ and $x \to g(x)$ be functions defined on an interval $[a, \omega)$ and integrable on every closed interval $[a, b] \subset [a, \omega)$. Suppose that g is monotonic. Then a sufficient condition for convergence of the improper integral

$$\int_a^{\omega}(fg)\,\mathrm{d}x$$

is that the one of the following pairs of conditions hold:

1. (a) the integral $\int_a^{\omega} f(x)\,\mathrm{d}x$ converges;
   (b) the function g is bound on $[a, \omega)$.
2. (a) the function $\mathcal{F}(b) = \int_a^b f(x)\,\mathrm{d}x$ is bound on $[a, \omega)$;
   (b) the integral $g(x)$ converges to zero as $x \to \omega, x \in [a, \omega)$.

4 Improper Integrals with More than one Sigularity

Examples 18. Let us discuss the integral

$$\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x$$

Examples 19. Let us discuss the **Euler-Possion** integral

$$\int_{-\infty}^{+\infty} e^{-x^2}\,\mathrm{d}x$$

Examples 20. Let us discuss the integral

$$\int_0^{+\infty}\frac{1}{x^{\alpha}}\,\mathrm{d}x$$

Examples 21. Let us discuss the integral

$$\int_0^{+\infty}\frac{\sin x}{x^{\alpha}}\,\mathrm{d}x$$

If
$$\mathrm{PV}\int_a^b f(x)\,\mathrm{d}x = \lim_{\delta\to 0+}\left[\int_a^{\omega-\delta} f(x)\,\mathrm{d}x + \int_{\omega+\delta}^b f(x)\,\mathrm{d}x\right] \tag{3}$$
the limits on the right hand side exist. This limit is called, following Cauchy, the principle value of the integral.