

# 曲线，曲面积分

武国宁\*

中国石油大学-北京

2017年8月

## 1 第一类曲线曲面积分

### 1.1 第一类曲线积分

#### 1.1.1 第一类曲线积分定义

**Definition 1** 设 $L$ 是空间 $\mathbb{R}^3$ 上可求长的连续曲线，其端点为 $A$ 和 $B$ ，函数 $f(x, y, z)$ 在 $L$ 上有界。令 $A = P_0, B = P_n$ ，在 $L$ 上顺次的插入分点 $P_1, P_2, \dots, P_{n-1}$ 。分别在每个小弧段上任意取一点 $(\xi_i, \eta_i, \zeta_i)$ ，并记第 $i$ 个弧段 $P_{i-1}P_i$ 的长度为 $\Delta s_i (i = 1, 2, \dots, n)$  做和式

$$\sum_{i=1}^n f(\xi_i, \eta_i, \zeta_i) \Delta s_i$$

如果当所有小弧段的最长长度趋于零时，这个和的极限存在且唯一，与分点和点的取法无关，则称该极限值为 $f(x, y, z)$ 在曲线 $L$ 上的第一类曲线积分，记为：

$$\int_L f(x, y, z) ds$$

即，

$$\int_L f(x, y, z) ds = \lim_{\lambda \rightarrow 0} \sum_{i=1}^n f(\xi_i, \eta_i, \zeta_i) \Delta s_i$$

---

\*电子邮件: wuguoning@163.com

其中 $f(x, y, z)$  称为被积函数,  $L$ 称为积分路径。

### 1.1.2 第一类曲线积分性质

**性质 1 (线性)** 如果函数 $f, g$ 在 $L$ 上的第一类曲线积分存在, 则对于任意常数 $\alpha \in \mathbb{R}, \beta \in \mathbb{R}$ , 有

$$\int_L (\alpha f + \beta g) ds = \alpha \int_L f ds + \beta \int_L g ds.$$

**性质 2 (路径可加性)** 设曲线 $L = L_1 + L_2$ , 如果函数 $f$ 在 $L$ 上的第一类曲线积分存在, 则函数 $f$ 在 $L_1$ 和 $L_2$ 上的积分也存在, 反之亦然。且有

$$\int_L f ds = \int_{L_1} f ds + \int_{L_2} f ds$$

**Theorem 1.1** 设函数 $f(x, y, z)$ 在 $L$ 上连续, 则它在 $L$ 上的第一类曲线积分存在, 且有

$$\int_L f(x, y, z) ds = \int_{\alpha}^{\beta} f(x(t), y(t), z(t)) \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt$$

**Example 1** 计算 $\int_L e^{\sqrt{x^2+y^2}} ds$ , 其中 $L$  为圆周  $x^2 + y^2 = a^2$ , 直线 $y = x$ 及 $x$ 轴在第一象限围城图形的边界。

**Example 2** 已知一条非均匀金属线 $L$ 的方程为

$$x = e^t \cos t, y = e^t \sin t, z = e^t, 0 \leq t \leq 1.$$

它在每一点的线密度与该点到原点的距离成反比, 而且在点 $(1, 0, 1)$ 处的线密度为 $1$ , 求它的质量。

**Example 3** 计算 $I = \int_L (x^2 + y^2 + z^2) ds$ , 其中 $L$ 为球面  $x^2 + y^2 + z^2 = a^2$ 和平面 $x + y + z = 0$ 的交线。

## 1.2 曲面的面积

### 1.2.1 曲面面积的计算

设曲面 $\Sigma$ 的方程为

$$x = x(u, v), y = y(u, v), z = z(u, v), (u, v) \in D$$

则

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k},$$

相应的Jacobi矩阵

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}$$

满秩, 考虑 $D$ 中的一个矩形微元 $\sigma$ , 它的四个顶点为:

$$P_1(u_0, v_0), P_2(u_0 + \Delta u, v_0), P_3(u_0 + \Delta u, v_0 + \Delta v), P_4(u_0, v_0 + \Delta v)$$

它被影射为:

$$Q_1 = \mathbf{r}(u_0, v_0), Q_2 = \mathbf{r}(u_0 + \Delta u, v_0), Q_3 = \mathbf{r}(u_0 + \Delta u, v_0 + \Delta v), Q_4 = \mathbf{r}(u_0, v_0 + \Delta v).$$

那么有:

$$\overrightarrow{Q_1Q_2} = \mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0) = \mathbf{r}_u(u_0, v_0)\Delta u + o(\Delta u),$$

$$\overrightarrow{Q_1Q_4} = \mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0) = \mathbf{r}_v(u_0, v_0)\Delta v + o(\Delta v),$$

$$\Delta S \approx \|\mathbf{r}_u(u_0, v_0) \times \mathbf{r}_v(u_0, v_0)\| \Delta u \Delta v.$$

$$dS = \|\mathbf{r}_u(u_0, v_0) \times \mathbf{r}_v(u_0, v_0)\| \Delta u \Delta v.$$

所以有,

$$\begin{aligned} S &= \iint_D \|\mathbf{r}_u(u_0, v_0) \times \mathbf{r}_v(u_0, v_0)\| \, du \, dv \\ &= \iint_D \sqrt{EG - F^2} \, du \, dv. \end{aligned}$$

其中,

$$\sqrt{EG - F^2} = \|\mathbf{r}_u \times \mathbf{r}_v\| = \left[ \frac{\partial(y, z)}{\partial(u, v)} \right]^2 + \left[ \frac{\partial(z, x)}{\partial(u, v)} \right]^2 + \left[ \frac{\partial(x, y)}{\partial(u, v)} \right]^2$$

现在考虑两种特殊情况:

1. 设曲面的方程为  $z = f(x, y)$ ,  $(x, y) \in D$  其中  $f(x, y)$  为连续可微函数,  $D$  为具有分段光滑边界的有界区域。这是有:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

这时有:

$$EG - F^2 = (1 + f_x^2)(1 + f_y^2) - (f_x f_y)^2 = 1 + f_x^2 + f_y^2.$$

于是有,

$$S = \iint_D \sqrt{1 + f_x^2(x, y) + f_y^2(x, y)} \, dx dy$$

2. 设曲面的方程为  $H(x, y, z) = 0$ , 其中  $H(x, y, z)$  是连续可微函数, 且在  $\Sigma$  上  $H_z(x, y, z) \neq 0$ . 由隐函数存在条件有:  $z = f(x, y)$ ,  $(x, y) \in D$  从而有:

$$\begin{aligned} S &= \iint_D \sqrt{1 + f_x^2 + f_y^2} \, dx dy \\ &= \iint_D \sqrt{1 + \left(-\frac{H_x}{H_z}\right)^2 + \left(-\frac{H_y}{H_z}\right)^2} \, dx dy \\ &= \iint_D \frac{\|\text{grad} H\|}{|H_z|} \, dx dy \end{aligned}$$

**Example 4** 求抛物面  $z = x^2 + y^2$  被平面  $z = 1$  所截出的部分的面积。

**Example 5** 设  $\Sigma$  为球面  $x^2 + y^2 + z^2 = 2Rz$  包含在锥面  $z^2 = 3(x^2 + y^2)$  内的部分, 求它的面积。

### 1.3 第一类曲面积分

设空间中一曲面  $\Sigma$  上分布着质量, 任意点  $(x, y, z)$  处的密度为  $\rho(x, y, z)$ , 如何求  $\Sigma$  的总质量。

**Definition 2** 设曲面 $\Sigma$ 为有界光滑（或分片光滑）曲面，函数 $z = f(x, y, z)$ 在曲面 $\Sigma$ 上有界。将曲面分成 $n$ 片小的曲面 $\Delta\Sigma_i, i = 1, 2, \dots, n$ ，记 $\Delta S_i$ 为第 $i$ 块曲面的面积，在 $\Delta\Sigma_i$ 上任取一点 $(\xi_i, \eta_i, \zeta_i)$ ，作和

$$\sum_{i=1}^n f(\xi_i, \eta_i, \zeta_i) \Delta S_i$$

如果当分割的细度趋于零时，这个和的极限存在且唯一，则称此极限为 $f(x, y, z)$ 在曲面 $\Sigma$ 上的第一类曲面积分，记为 $\iint_{\Sigma} f(x, y, z) dS$ 。记为

$$\iint_{\Sigma} f(x, y, z) dS = \lim_{\lambda \rightarrow 0} \sum_{i=1}^n f(\xi_i, \eta_i, \zeta_i) \Delta S_i,$$

其中 $\Sigma$ 为积分曲面， $f(x, y, z)$ 为被积函数。

### 1.3.1 第一类曲面积分的计算方法

设 $\Sigma$ 的方程为：

$$x = x(u, v), y = y(u, v), z = z(u, v), (u, v) \in D.$$

$f(x, y, z)$ 在 $\Sigma$ 上连续，则有：

$$\iint_{\Sigma} f(x, y, z) dS = \iint_D f(x(u, v), y(u, v), z(u, v)) \sqrt{EG - F^2} du dv$$

特别的，当曲面 $\Sigma$ 的方程为 $z = z(x, y), (x, y) \in D$ ，则有

$$\iint_{\Sigma} f(x, y, z) dS = \iint_D f(x, y, z(x, y)) \sqrt{1 + z_x^2(x, y) + z_y^2(x, y)} dx dy$$

**Example 6** 计算 $I = \iint_{\Sigma} \sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}} dS$  其中， $\Sigma$ 为椭球面 $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, a, b, c > 0$ .

## 2 第二类曲线，曲面积分

### 2.1 第二类曲线积分

#### 2.1.1 Vector Fields

Suppose a region in the plane or in the space is occupied by a moving fluid such as air or water. Imaging that the fluid is made up of a very large number of particles, and that any instant of time a particle has a velocity  $\mathbf{v}$ . If we take a picture of some particles at different position points at the same instant, we would expect to find that these velocities vary from position to position. We can think of a velocity vectors as being attached to each point of the fluid. Such a fluid exemplifies a **vector field**. Generally, a vector field on a domain in the plane or

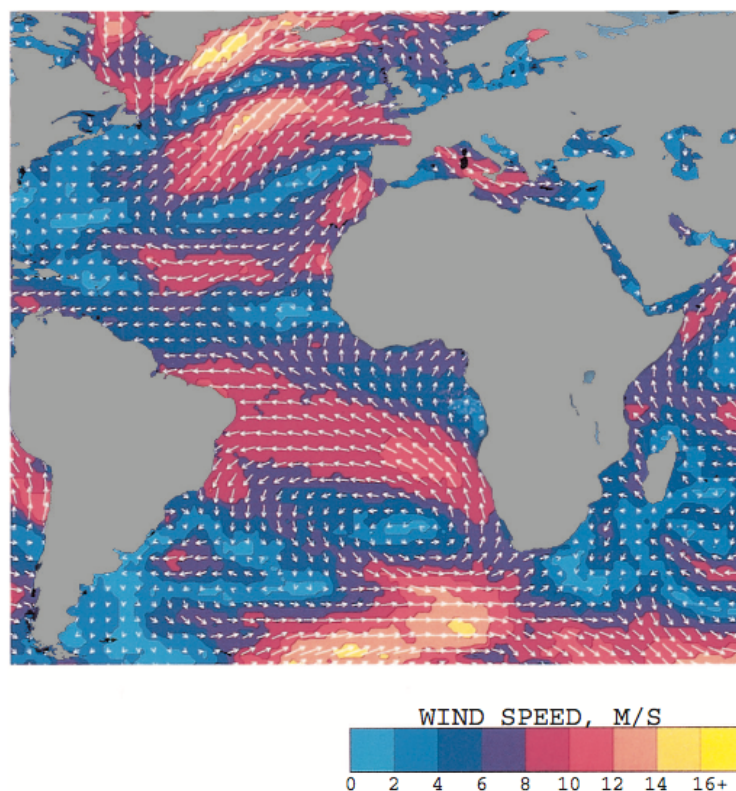


Figure 1: NASA's Seasat used radar to take 350,000 wind measurement over the world's oceans. The arrows show wind direction; their length and the color contouring indicate speed. Notice the heavy storm of Greenland.

in space is a function that assigns a vector to each point in the domain. A field of

three-dimensional vectors might have a formula like

$$\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}.$$

### 2.1.2 The Work of a Field

Let  $\mathbf{F}(x)$  be a continuous force field acting in a domain  $G$  of the Euclidean space  $\mathbb{R}^n$ . The displacement of a test particle in the field is accompanied by work. We ask how we can compute the work done by the field in moving a unit test particle along a given trajectory, more precisely, a smooth path  $\gamma : I(\gamma) \subset G$ . It is known that in a constant field  $\mathbf{F}$  the displacement by a vector  $\boldsymbol{\xi}$  is associated with an amount of work  $\langle \mathbf{F}, \boldsymbol{\xi} \rangle$

Suppose that the vector field

$$\mathbf{F}(\mathbf{t}) = M(x(t), y(t), z(t))\mathbf{i} + N(x(t), y(t), z(t))\mathbf{j} + P(x(t), y(t), z(t))\mathbf{k}$$

represents a force throughout a region in space (it might be the force of gravity or an electromagnetic force of some kind), and  $t \rightarrow \mathbf{r}(t)$  be a smooth mapping:  $\gamma : I \rightarrow G$  defined on the closed interval  $I = \{t \in \mathbb{R} | a \leq t \leq b\}$

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, a \leq t \leq b,$$

is a smooth curve in the region.

We take a sufficiently fine partition of the closed interval  $[a, b]$ . Then on each interval  $I_i = \{t \in I | t_{i-1} \leq t \leq t_i\}$  of the partition we have the equality

$$\Delta \mathbf{r}_i = \mathbf{r}_{i+1} - \mathbf{r}_i \approx \dot{\mathbf{r}}(t_i)\Delta t_i = (\dot{x}(t_i), \dot{y}(t_i), \dot{z}(t_i)) \Delta t_i.$$

Since the field  $\mathbf{F}(t)$  is continuous, it can be regarded as locally constant, and for that reason we can compute the work  $\Delta A_i$  as

$$\Delta A_i \approx \langle \mathbf{F}(t_i), \dot{\mathbf{r}}(t_i)\Delta t_i \rangle.$$

$$A = \sum_i \Delta A_i \approx \sum_i \langle \mathbf{F}(t_i), \dot{\mathbf{r}}(t_i)\Delta t_i \rangle.$$

and so passing to the limit as the partition of the closed interval  $I$  is refined, we find that

$$A = \int_a^b \langle \mathbf{F}(t), \dot{\mathbf{r}}(t) \rangle dt. \quad (1)$$

The expression  $\langle \mathbf{F}(t), \dot{\mathbf{r}}(t) \rangle dt$  is written as  $\langle \mathbf{F}(t), d\mathbf{r} \rangle$ , then as assume the coordinates in  $\mathbb{R}^3$  are Cartesian coordinates, we can give this expression the form

$$Mdx + Ndy + Pdz,$$

after which we can write Eq. 1 as

$$A = \int_{\gamma} Mdx + Ndy + Pdz \quad (2)$$

or as

$$A = \int_{\gamma} \omega_{\mathbf{F}}^1 \quad (3)$$

Formular 3 provides the precise meaning of the integrals of the work 1-form along the path  $\gamma$ .

The expression of Equation 2 can also be written as

$$A = \int_{\gamma} \mathbf{F} \cdot \mathbf{T} ds \quad (4)$$

where  $\mathbf{T} = (\cos \alpha, \cos \beta, \cos \gamma)$  is the unit tangent vector.

**Example 7** Consider the force field  $\mathbf{F} = \left( -\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$  defined at all points of the plane  $\mathbb{R}^2$  except the origin. Let us compute the work of this field along the curve  $\gamma_1$  defined as  $x = \cos t, y = \sin t, 0 \leq t \leq 2\pi$ , and along the curve defined by  $x = 2 + \cos t, y = \sin t, 0 \leq t \leq 2\pi$

**Example 8** Let  $\mathbf{r}$  be the radius vector of a point  $(x, y, z) \in \mathbb{R}^3$  and  $r = |\mathbf{r}|$ . Suppose a force field  $\mathbf{F} = f(r)\mathbf{r}$  is defined everywhere in  $\mathbb{R}^3$  except at the origin. This is so-called central force field. Let us find the work of  $\mathbf{F}$  on a path:  $\gamma$  :



$[0, 1] \rightarrow \mathbb{R}^3 \setminus 0$

$$\begin{aligned}\int_{\gamma} f(r)(x dx + y dy + z dz) &= \frac{1}{2} \int_{\gamma} f(r) d(x^2 + y^2 + z^2) = \frac{1}{2} \int_0^1 f(r(t)) dr^2(t) \\ &= \frac{1}{2} \int_0^1 f(\sqrt{u(t)}) du(t) = \frac{1}{2} \int_{r_0^2}^{r_1^2} f(\sqrt{u}) du \\ &= \Phi(r_0, r_1).\end{aligned}$$

In particular, for the gravitational field  $\frac{1}{r^3} \mathbf{r}$  of a unit point mass located at the origin, we obtain

$$\Phi(r_0, r_1) = \frac{1}{2} \int_{r_0^2}^{r_1^2} \frac{1}{u^{\frac{3}{2}}} du = \frac{1}{r_0} - \frac{1}{r_1}$$

**Example 9** Find the work done by a variable force over a space curve, where the force is  $\mathbf{F} = (y - x^2)\mathbf{i} + (z - y^2)\mathbf{j} + (x - z^2)\mathbf{k}$  over the curve  $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ , from  $(0, 0, 0)$  to  $(1, 1, 1)$ .

**Example 10** Find flow along a helix: A fluid's velocity field is  $\mathbf{F} = x\mathbf{i} + z\mathbf{j} + k\mathbf{k}$ . Find the flow along the helix  $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$

### 2.1.3 Flux Across a Plane Curve

To find the rate at which a fluid is entering or leaving a region enclosed by a smooth curve  $C$  in the xy-plane, we calculate the line integral over  $C$  of  $\mathbf{F} \cdot \mathbf{n}$ , the scalar component of the fluid's velocity field in the direction of the curve's outward-pointing normal vector. The value of this integral is the flux <sup>1</sup> of  $\mathbf{F}$  across  $C$ .

**Definition 3** If  $C$  is a smooth curve in the domain of a continuous vector field  $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$  in the plane and if  $\mathbf{n}$  is the outward-pointing unit normal vector on  $C$ , the flux of  $\mathbf{F}$  across  $C$  is :

$$A = \int_C \mathbf{F} \cdot \mathbf{n} ds \tag{5}$$

---

<sup>1</sup>Flux is a Latin word for flow, but many flux calculation involve no motion at all.

To evaluate the integral of Equation 5, we begin with a smooth parameterization

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}, a \leq t \leq b.$$

If the motion is counterclockwise, then

$$\mathbf{n} = \mathbf{T} \times \mathbf{k},$$

and if the motion is clockwise, then

$$\mathbf{n} = -\mathbf{T} \times \mathbf{k},$$

where  $\mathbf{T} = \frac{\dot{\mathbf{r}}(t)}{\|\dot{\mathbf{r}}(t)\|}$ . So for counterclockwise motion, the calculation of Equation 5 is:

$$\begin{aligned} A &= \oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_a^b \mathbf{F} \cdot \frac{\dot{\mathbf{r}}(t) \times \mathbf{k}}{\|\dot{\mathbf{r}}(t)\|} \|\dot{\mathbf{r}}(t)\| \, dt \\ &= \int_a^b \mathbf{F} \cdot \dot{\mathbf{r}}(t) \times \mathbf{k} \, dt = \oint_C M \, dy - N \, dx \end{aligned} \tag{6}$$

**Example 11** Find the flux of  $\mathbf{F} = (x - y)\mathbf{i} + x\mathbf{j}$  across the circle  $x^2 + y^2 = 1$  in the  $xy$ -plane.

**(Method I)** Parametrize the circle:  $x = \cos t, y = \sin t, 0 \leq t \leq 2\pi$

$$\begin{aligned} A &= \oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_0^{2\pi} (\cos t - \sin t, \cos t) \cdot (\cos t, \sin t) \, dt \\ &= \int_0^{2\pi} \cos^2 t \, dt = \pi \end{aligned}$$

**(Method II)**

$$A = \oint_C M \, dy - N \, dx = \int_0^{2\pi} \cos t - \sin t \, d \sin t - \cos t \, d \cos t = \pi$$

### 3 Surface Area and Surface Integrals

We know how to integrate a function over a flat region in a plane, but what if the function is defined over a curved surface? To evaluate one of these so-called surface integrals, we rewrite it as double integral over a region in a coordinate plane beneath the surface.

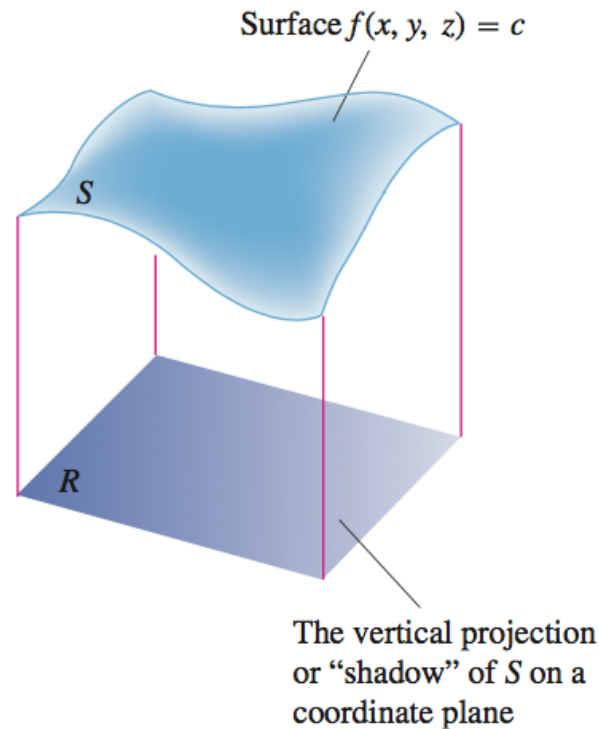


Figure 2: As we soon see, the integral of a function  $g(x, y, z)$  over a surface  $S$  in space can be calculated by evaluating a related double integral over the vertical projection or "shadow" of  $S$  on a coordinate plane.

#### 3.1 Surface Area

Figure 3 shows a surface  $S$  lying above its "shadow" region  $R$  in a plane beneath it. The surface is defined by the equation  $f(x, y, z) = c$ . If the surface is smooth ( $\nabla f$  is continuous and never vanishes on  $S$ ). We can define and calculate its area as a double integral over  $R$ .

The first step in defining the area of  $S$  is to partition the region  $R$  into small rectangles  $\Delta A_k$  of the kind we would use if we were defining an integral over  $R$ .

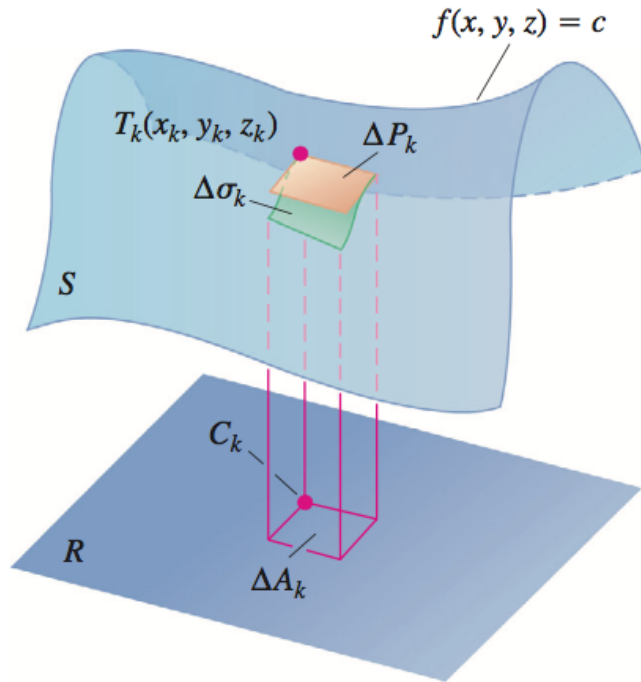


Figure 3: A surface  $S$  and its vertical projection onto a plane beneath it. You can think of  $R$  as the shadow of  $S$  on the plane. The tangent plane  $\Delta P_k$  approximates the surface patch  $\Delta\sigma_k$  above  $\Delta A_k$ .

Directly above each  $\Delta A_k$  lies a patch of surface  $\Delta\sigma_k$  that we may approximate by a parallelogram  $\Delta P_k$  in the tangent plane to  $S$  at a point  $T_k(x_k, y_k, z_k)$  in  $\Delta\sigma_k$ .

Figure 4 give a magnified view of  $\Delta\sigma_k$  and  $\Delta P_k$ , showing the gradient vector  $\nabla f_k$  at  $T_k$  and a unit vector  $\mathbf{p}$  that is normal to  $R$ . The figure also shows the angle  $\gamma_k$  between  $\nabla f_k$  and  $\mathbf{p}$ . In our case, this translates into the statement

$$|(\mathbf{u}_k \times \mathbf{v}_k) \cdot \mathbf{p}| = \Delta A_k$$

or

$$\Delta P_k |\cos \gamma_k| = \Delta A_k$$

or

$$\Delta P_k = \frac{\Delta A_k}{|\cos \gamma_k|}$$

We will have  $\cos \gamma_k \neq 0$  if  $\nabla f$  is not parallel to the ground plane and  $\nabla f \cdot \mathbf{p} \neq 0$ .

Since the patches  $\Delta P_k$  approximates the surface patches  $\Delta\sigma_k$  that fit together

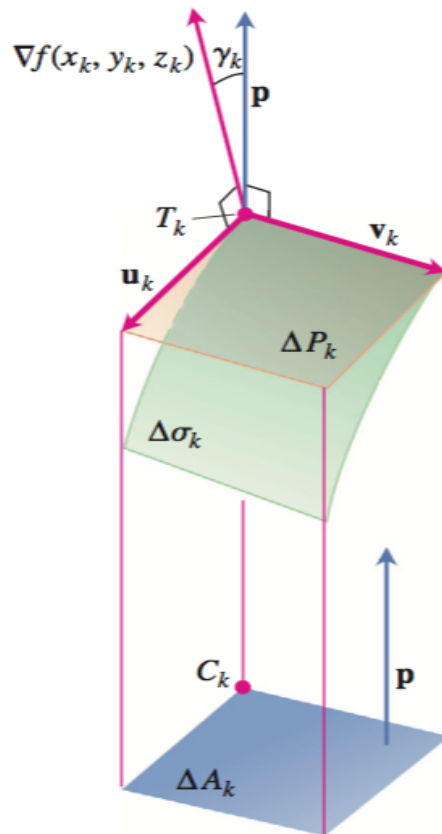


Figure 4: Magnified view from the preceding figure. The vector  $\mathbf{u}_k \times \mathbf{v}_k$  is parallel to the vector  $\nabla f$  because both vectors are normal to the plane of  $\Delta P_k$ .

to make  $S$ , the sum

$$\sum \Delta P_k = \sum \frac{\Delta A_k}{|\cos \gamma_k|} \quad (7)$$

If we refined the partition of  $R$ . In fact, the sums on the right-hand side of the equation 7 are approximating sums for the double integral.

$$\iint_R \frac{1}{|\cos \gamma|} dA. \quad (8)$$

We therefore define the area of  $S$  to be the value of this integral whenever it exists.

For any surface  $f(x, y, z) = c$ , we have  $|\nabla f \cdot \mathbf{p}| = |\nabla f| |\mathbf{p}| |\cos \gamma|$ , so

$$\frac{1}{|\cos \gamma|} = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}.$$

The area of the surface  $f(x, y, z) = c$  over a closed and bounded plane region  $R$  is

$$S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA. \quad (9)$$

**Example 12** Find the area of the surface cut from the bottom of the paraboloid  $x^2 + y^2 - z = 0$  by the plane  $z = 4$ .

We have

$$\begin{aligned} f(x, y, z) &= x^2 + y^2 - z, \nabla f = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}, \\ |\nabla f \cdot \mathbf{p}| &= |\nabla f \cdot \mathbf{k}| = 1. \\ S &= \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dx dy = \iint_{x^2+y^2 \leq 4} \sqrt{4x^2 + 4y^2 + 1} dx dy. \end{aligned}$$

**Example 13** Find the area of the cap cut from the hemisphere  $x^2 + y^2 + z^2 = 2$ , by the cylinder  $x^2 + y^2 \leq 1$  in the  $xy$ -plane.

### 3.2 Surface Integrals

Suppose, for example, that we have an electrical charge distributed over a surface  $f(x, y, z) = c$  like the one shown in Figure 3 and that the function  $g(x, y, z)$  gives the charge per unit area (charge density) at each point on  $S$ . Then we may

calculate the total charge on  $S$  as an integral of below:

$$\text{Totalcharge} \approx \sum g(x_k, y_k, z_k) \Delta P_k = \sum g(x_k, y_k, z_k) \frac{\Delta A_k}{|\cos \gamma|}$$

If  $f$ , the function defining the surface  $S$ , and its first derivatives are continuous, and if  $g$  is continuous over  $S$ , then the sums on the right-hand side of the last equation approach the limit

$$\iint_R g(x, y, z) \frac{dA}{|\cos \gamma|} = \iint_R g(x, y, z) \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA. \quad (10)$$

**The Surface Area Differential and the Differential Form for Surface Integrals**

$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}$$

$$\iint_S g d\sigma$$

**Example 14** Integrate  $g(x, y, z) = xyz$  over the surface of the cube cut from the first octant by the planes  $x = 1, y = 1, z = 1$  (Figure 5)

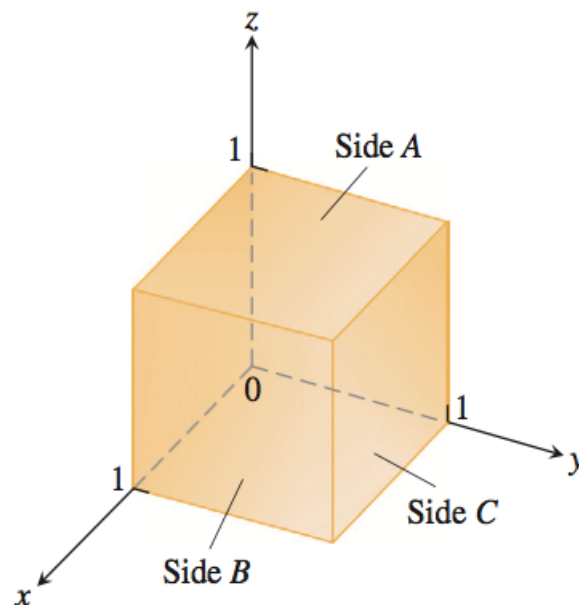


Figure 5: The cube in example above.

### 3.3 Orientation

We call a smooth surface  $S$  **orientable** or **two-sided** if it is possible to define a field  $\mathbf{n}$  of unit normal vectors on  $S$  that varies continuously with position. Once  $\mathbf{n}$  has been chosen, we say that we have **oriented** the surface, and we call the surface together with its normal field an **oriented surface**. The vector  $\mathbf{n}$  at any point is called the **positive direction** at that point.

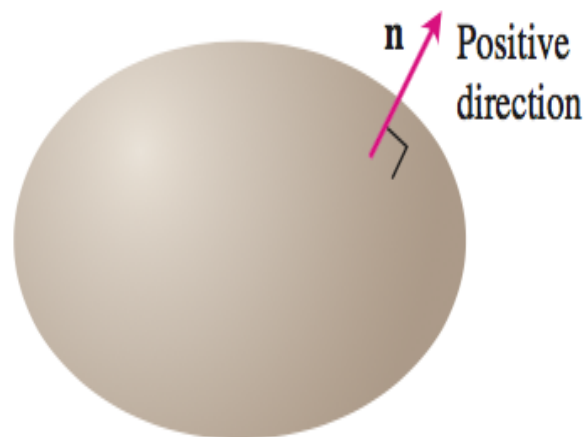


Figure 6: Smooth closed surface in space is orientable. The outward unit normal vector defines the positive direction at each point.

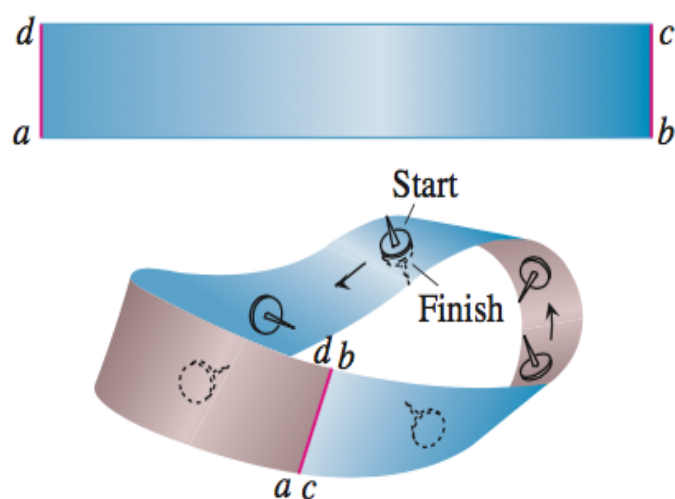


Figure 7: The Möbius band is a non-orientable or one-side surface.



### 3.4 Surface Integral for Flux

Suppose that  $\mathbf{F}$  is a continuous vector field defined over an oriented surface  $S$  and that  $\mathbf{n}$  is the chosen unit normal field on the surface. We call the integral of  $\mathbf{F} \cdot \mathbf{n}$  over  $S$  the flux of  $\mathbf{F}$  across  $S$  in the positive direction.

**Definition 4** The **flux** of a three-dimensional vector field  $\mathbf{F}$  across an oriented surface  $S$  in the direction of  $\mathbf{n}$  is

$$\text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma$$

The definition is analogous to the flux of a two-dimensional field  $\mathbf{F}$  across a plane curve  $C$ . In the plane, the flux is

$$\int_C \mathbf{F} \cdot \mathbf{n} \, ds,$$

the integral of the scalar component of  $\mathbf{F}$  normal to the curve.

If  $\mathbf{F}$  is the velocity field of a three-dimensional fluid flow, the flux of  $\mathbf{F}$  across surface  $S$  is the net rate at which fluid is crossing  $S$  in the chosen positive direction. If  $S$  is part of a surface  $g(x, y, z) = c$ , then  $\mathbf{n}$  may be taken to be one of the two fields

$$\mathbf{n} = \pm \frac{\nabla g}{|\nabla g|} \quad (11)$$

depending on which one gives the preferred direction. The corresponding flux is

$$\begin{aligned} \text{Flux} &= \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma \\ &= \iint_S \left( \mathbf{F} \cdot \frac{\pm \nabla g}{|\nabla g|} \right) \frac{|\nabla g|}{|\nabla g \cdot \mathbf{p}|} \, dA \\ &= \iint_S \mathbf{F} \cdot \frac{\nabla g}{|\nabla g \cdot \mathbf{p}|} \, dA \end{aligned} \quad (12)$$

**Example 15** Find the flux of  $\mathbf{F} = yz\mathbf{j} + z^2\mathbf{k}$  outward through the surface  $S$  cut from the cylinder  $y^2 + z^2 = 1, z \geq 0$  by the plane  $x = 0$  and  $x = 1$ . See Figure 6.

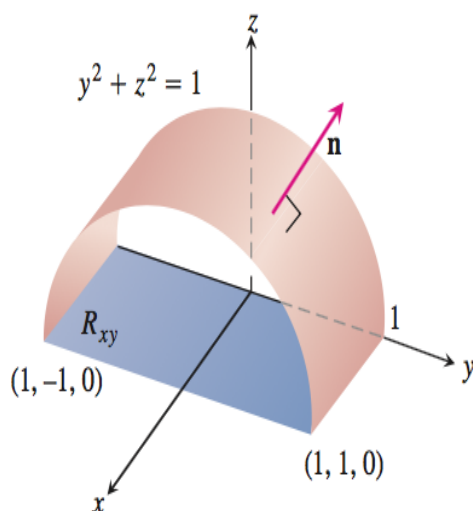


Figure 8: Calculating the flux of a vector field outward through this surface. The area of the shadow region  $R_{xy}$  is 2.

**Solution:**

The outward normal field on  $S$  may be calculated from the gradient of  $g(x, y, z) = y^2 + z^2$  to be

$$\mathbf{n} = + \frac{\nabla g}{|\nabla g|} = \frac{2y\mathbf{j} + 2z\mathbf{k}}{\sqrt{4y^2 + 4z^2}} = \frac{2y\mathbf{j} + 2z\mathbf{k}}{2} = y\mathbf{j} + z\mathbf{k}.$$

With  $\mathbf{p} = \mathbf{k}$ , we also have

$$d\sigma = \frac{|\nabla g|}{|\nabla g \cdot \mathbf{k}|} dA = \frac{2}{|2z|} dA = \frac{1}{z} dA.$$

$$\mathbf{F} \cdot \mathbf{n} = (yz\mathbf{j} + z^2\mathbf{k}) \cdot (y\mathbf{j} + z\mathbf{k}) = z(y^2 + z^2) = z.$$

$$\iint_S \mathbf{F} \cdot \mathbf{n} d\sigma = \iint_S z \frac{1}{z} dA = \iint_{R_{xy}} dA = 2.$$

### 3.5 Parametrized Surfaces

We have defined curves in the plane in three different ways:

Explicit form:  $y = f(x)$ ,

Implicit form:  $F(x, y) = 0$ ,

Parametric vector form:  $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ .

We have analogous definition of surface in space:

Explicit form:  $z = f(x, y)$ ,

Implicit form:  $F(x, y, z) = 0$ .

There is also a parametric form that gives the position of a point on the surface as a vector function of two variables. The present section extends the investigation of surface area and surface integrals to surface described parametrically.

### 3.5.1 Parametrizations of Surfaces

Let

$$\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k} \quad (13)$$

be a continuous vector function that is defined on a region  $R$  in the  $uv$ -plane.

**Example 16** *Find a parametrization of the cone*

$$z = \sqrt{x^2 + y^2}, 0 \leq z \leq 1.$$

**Example 17** *Find a parametrization of the sphere*

$$x^2 + y^2 + z^2 = a^2$$

**Example 18** *Find a parametrization of the cylinder*

$$x^2 + (y - 3)^2 = 9, 0 \leq z \leq 5.$$

### 3.5.2 Surface Area

For details see **section 1.2.1**.

### 3.5.3 Surface Integral

For detail see **section 1.3**

### 3.5.4 Flux

**Example 19** Find the flux of  $\mathbf{F} = yz\mathbf{i} + x\mathbf{j} - z^2\mathbf{k}$  outward through the parabolic cylinder  $y = x^2$ ,  $0 \leq x \leq 2$ ,  $0 \leq z \leq 4$ .

## 4 Path Independence, Potential Functions, and Conservative Fields

In gravitational and electric fields, the amount of work it takes to move a mass or a charge from one point to another depends only on the object's initial and final positions and not on the path taken in between.

### 4.1 Path Independence

If  $A$  and  $B$  are two points in an open region  $D$  in space, the work  $\int \mathbf{F} \cdot d\mathbf{r}$  done in moving a particle from  $A$  to  $B$  by a field  $\mathbf{F}$  defined on  $D$  usually depends on the path taken. For some special field, however, the integral's value is the same for all paths from  $A$  to  $B$ .

**Definition 5** Let  $\mathbf{F}$  be a field defined on an open region  $D$  in space, and suppose that for any two points  $A$  and  $B$  the work  $\int_A^B \mathbf{F} \cdot d\mathbf{r}$  done in moving from  $A$  to  $B$  is the same over all paths from  $A$  to  $B$ . Then the integral  $\int \mathbf{F} \cdot d\mathbf{r}$  is path independent in  $D$  and the field  $\mathbf{F}$  is conservative on  $D$ .

Under differentiability conditions normally met in practice, a field  $\mathbf{F}$  is conservative if and only if it is the gradient of a scalar function, that is, if and only if  $\mathbf{F} = \nabla f$  for some  $f$ . The function  $f$  is called the potential function.

**Definition 6** If  $\mathbf{F}$  is a field defined on  $D$  and  $\mathbf{F} = \nabla f$  for some scalar function  $f$  on  $D$ , then  $f$  is called a **potential function for  $\mathbf{F}$** .

### 4.2 Line Integrals in Conservative Fields

**Theorem 4.1 The Fundamental Theorem of Line Integral**

Let  $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$  be a vector field whose component are continuous throughout an open connected  $D$  in space. Then there exists a differentiable function  $f$  such that

$$\mathbf{F} = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

if and only if for all points  $A$  and  $B$  in  $D$  the value of  $\int_A^B \mathbf{F} \cdot d\mathbf{r}$  is independent of the path joining  $A$  to  $B$  in  $D$ .

If the integral is independent of the path from  $A$  to  $B$ , its value is

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A).$$

**Example 20** Find the work done by the conservative field

$$\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$$

along any smooth curve  $C$  joining the point  $A(-1, 3, 9)$  to  $B(1, 6, -4)$ .

**Theorem 4.2 Closed-Loop Property of Conservative Fields** The following statements are equivalent.

1.  $\int \mathbf{F} \cdot d\mathbf{r} = 0$  around every closed loop in  $D$ .
2. The field  $\mathbf{F}$  is conservative on  $D$ .

### 4.3 Find Potentials for Conservative Fields

**Theorem 4.3** Suppose that the domain of  $\mathbf{F}$  is connected and simply connected. Let  $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$  be a vector field whose component have continuous first partial derivatives. Then,  $\mathbf{F}$  is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y},$$

**Example 21** Show that  $\mathbf{F} = (e^x \cos y + yz)\mathbf{i} + (xz - e^x \sin y)\mathbf{j} + (yz + z)\mathbf{k}$  is conservative and find a potential function for it.

## 5 Green's Theorem in the Plane

In this section we consider how to evaluate the integral if it is not associated with a conservative vector field, but is a flow or flux integral across a closed curve in the  $xy$ -plane.

### 5.1 Divergence

We need new ideas for Green's theorem. The first is the idea of the divergence of a vector field at a point, sometimes called the flux density of the vector field by physicists and engineers.

Suppose that  $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$  is the velocity field of a fluid flow in the plane and that the first partial derivatives of  $M$  and  $N$  are continuous at each point of a region  $R$ . Let  $(x, y)$  be a point in  $R$  and let  $A$  be a small rectangle with one corner at  $(x, y)$  that, along with its interior, lies entirely in  $R$ . The sides of the rectangle parallel to the coordinate axes, have lengths of  $\Delta x$  and  $\Delta y$ . The rate at which fluid leaves the rectangle across the bottom edge is approximately, see Figure 9

$$\mathbf{F}(x, y) \cdot -\mathbf{j}\Delta x = -N(x, y)\Delta x$$

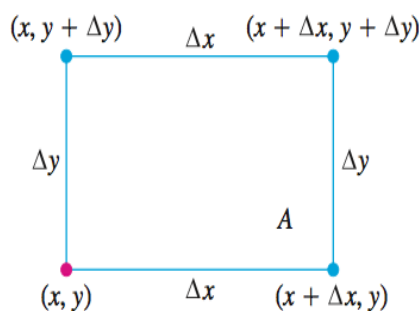


Figure 9: The rectangle for defining the divergence (flux density) of a vector field at point  $(x, y)$ .

Exit Rates:

1. Top:  $\mathbf{F}(x, y + \Delta y) \cdot \mathbf{j}\Delta x = N(x, y + \Delta y)\Delta x$

2. Bottom:  $\mathbf{F}(x, y) \cdot -\mathbf{j}\Delta x = -N(x, y)\Delta x$
3. Right:  $\mathbf{F}(x + \Delta x, y) \cdot \mathbf{i}\Delta y = M(x + \Delta x, y)\Delta y$
4. Left:  $\mathbf{F}(x, y) \cdot -\mathbf{i}\Delta y = -M(x, y)\Delta y$ .

Combining opposite pairs gives:

$$\text{Flux across rectangle boundary} \approx \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) \Delta x \Delta y.$$

We now divide by  $\Delta x \Delta y$  to estimate the total flux per unit area of flux density for the rectangle:

$$\text{div}\mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$$

**Definition 7** *The divergence (flux density) of a vector field  $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$  at the point  $(x, y)$  is*

$$\text{div}\mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$$

**Example 22** *Find the divergence of  $\mathbf{F}(x, y) = (x^2 - y)\mathbf{i} + (xy - y^2)\mathbf{j}$ .*

## 5.2 Spin Around an Axis: The k-Component of Curl

The second idea we need for Green's theorem has to do with measuring how a paddle wheel spins at a point in a fluid flowing in a plane region. This idea gives some sense of how the fluid is circulating around axes located at different points and perpendicular to the region. Physicists sometimes refer to this as the circulation density of a vector field  $\mathbf{F}$  at a point. To obtain it, we return to the velocity field.

$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$

and the rectangle  $A$ . The rectangle is redrawn here as Figure 10.

The counterclockwise circulation of the velocity  $\mathbf{F}$  around the boundary of  $A$  is the sum of flow rates along the sides.

1. Top:  $\mathbf{F}(x, y + \Delta y) \cdot -\mathbf{i}\Delta x = -M(x, y + \Delta y)\Delta x$

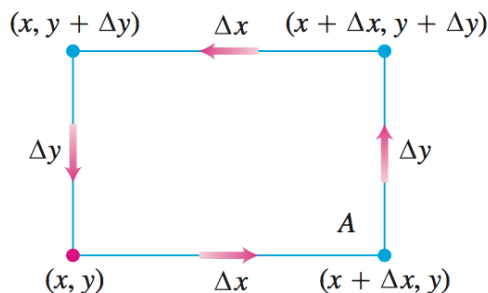


Figure 10: The rectangle for defining the curl (circulation density) of a vector field at point  $(x, y)$ .

2. Bottom:  $\mathbf{F}(x, y) \cdot \mathbf{i}\Delta x = M(x, y)\Delta x$
3. Right:  $\mathbf{F}(x + \Delta x, y) \cdot \mathbf{j}\Delta y = N(x + \Delta x, y)\Delta y$
4. Left:  $\mathbf{F}(x, y) \cdot -\mathbf{j}\Delta y = -N(x, y)\Delta y$ .

We add opposite pairs to get:

$$\text{Circulation along the boundary} \approx \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \Delta x \Delta y.$$

**Definition 8** The  $k$ -component of the curl (circulation density) of a vector field  $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$  at the point  $(x, y)$  is the scalar

$$\text{curl}\mathbf{F} \cdot \mathbf{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

**Example 23** Find the  $k$ -component of the curl for the vector field

$$\mathbf{F}(x, y) = (x^2 - y)\mathbf{i} + (xy - y^2)\mathbf{j}$$

### 5.3 Two Forms for Green's Theorem

In one form, Green's Theorem says that under suitable conditions the outward flux of a vector field across a simple closed curve in the plane equals the double integral of the divergence of the field over the region enclosed by the curve.

#### Theorem 5.1 Flux-Divergence or Normal Form

The outward flux of a field  $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$  across a simple closed curve  $C$  equals



the double integral of  $\text{div } \mathbf{F}$  over the region  $R$  enclosed by  $C$ .

$$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \oint_C M \, dy - N \, dx = \iint_R \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) \, dx \, dy \quad (14)$$

In another form, Green's Theorem says that the counterclockwise circulation of a vector around a simple closed curve is the double integral of the  $k$ -component of the curl of the field over the region enclosed by the curve.

**Theorem 5.2 (Circulation-Curl or Tangential Form)** *The counterclockwise circulation of a field  $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$  around a simple closed curve  $C$  in the plane equals the double integral of  $(\text{curl } \mathbf{F}) \cdot \mathbf{k}$  over the region  $R$  enclosed by  $C$ .*

$$\oint_C \mathbf{F} \cdot \mathbf{T} \, ds = \oint_C M \, dx + N \, dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dx \, dy. \quad (15)$$

**Example 24** *Verify both forms of Green's Theorem for the field*

$$\mathbf{F}(x, y) = (x - y)\mathbf{i} + x\mathbf{j}$$

and the region  $R$  bounded by the unit circle

$$C : \mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j}, 0 \leq t \leq 2\pi.$$

**Example 25** *Evaluate the integral*

$$\oint_C xy \, dy - y^2 \, dx$$

where  $C$  is the square cut from the first quadrant by the lines  $x = 1$  and  $y = 1$ .

**Example 26** *Calculate the outward flux of the field  $\mathbf{F}(x, y) = x\mathbf{i} + y^2\mathbf{j}$  across the square bounded by the lines  $x = \pm 1$  and  $y = \pm 1$ .*

**Example 27** *Verify the circulation form of Green's Theorem on the annular ring  $R : h^2 \leq x^2 + y^2 \leq 1, 0 < h < 1$ , if*

$$M = \frac{-y}{x^2 + y^2}, N = \frac{x}{x^2 + y^2}$$

Here  $R$  is not simply connected.

## 6 The Divergence Theorem and a Unified Theory

The divergence form of Green's Theorem in the plane states that the net outward flux of the field across a simple closed curve can be calculated by integrating the divergence of the field over the region enclosed by the curve. The corresponding theorem in three dimensions, called the Divergence Theorem, states that the net outward flux of a vector field across a closed surface in space can be calculated by integrating the divergence of the field over the region enclosed by the surface.

### 6.1 Divergence in Three dimension

The divergence of a vector field  $\mathbf{F} = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$  is the scalar function

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}.$$

**Example 28** Find the divergence of  $\mathbf{F} = 2xz\mathbf{i} - xy\mathbf{j} - z\mathbf{k}$ .

### 6.2 The Gauss-Ostrogradskii Formula

**Theorem 6.1** Let  $\mathbb{R}^3$  be three-dimensional space with a fixed coordinates system  $x, y, z$  and  $\bar{D}$  a compact domain in  $\mathbb{R}^3$  bounded by piecewise-smooth surface. Let  $P, Q, R$  be smooth functions in the closed domain  $\bar{D}$ . Then the following relation holds:

$$\begin{aligned} & \iiint_{\bar{D}} \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx dy dz \\ &= \iint_{\partial \bar{D}} P dy \wedge dz + Q dz \wedge dx + R dx \wedge dy. \end{aligned}$$

If we denote  $\mathbf{F} = (P, Q, R)$ , the Gauss-Ostrogradskii theorem can be written as

$$\iiint_{\bar{D}} \nabla \cdot \mathbf{F} dV = \iint_{\partial \bar{D}} \mathbf{F} \cdot \mathbf{n} d\sigma.$$

**Example 29** Find the flux of  $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$  outward through the surface of the cube cut from the first octant by the plane  $x = 1, y = 1, z = 1$ .

**Example 30** Find the net outward flux of the field

$$\mathbf{F} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\rho^3}, \rho = \sqrt{x^2 + y^2 + z^2}$$

across the boundary of the region:  $D : 0 < a^2 \leq x^2 + y^2 + z^2 \leq b^2$

### 6.3 Gauss's Law: One of the Four Great Laws of Electromagnetic Theory

The outward flux of  $\mathbf{E}$  across any sphere centered at the origin is  $\frac{q}{\epsilon_0}$ , and the result is not confined to spheres. In electro-magnetic theory, the electric field created by a point charge  $q$  located at the origin is

$$\mathbf{E}(x, y, z) = \frac{1}{4\pi\epsilon_0} \frac{q}{|\mathbf{r}|^2} \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{q}{4\pi\epsilon_0} \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\rho^3}$$

Gauss's law:

$$\iint_S \mathbf{E} \cdot \mathbf{n} d\sigma = \frac{q}{\epsilon_0}$$

## 7 Stokes' Theorem

In three dimensional, the circulation around a point  $P$  in a plane is described with a vector. This vector is normal to the plane of the circulation and points in the direction that gives it a right-hand relation to the circulation line. The length of the vector gives the rate of the fluid's rotation, which usually varies as the circulation plane is tilted about  $P$ . It turns out that the vector of greatest circulation in a flow with velocity field  $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$  is the curl vector

$$\text{curl}\mathbf{F} = \left( \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) \mathbf{i} + \left( \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x} \right) \mathbf{j} + \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k}. \quad (16)$$

If we denote

$$\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}. \quad (17)$$

The curl of  $\mathbf{F}$  is  $\nabla \times \mathbf{F}$ :

$$\begin{aligned} \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} \\ &= \left( \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) \mathbf{i} + \left( \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x} \right) \mathbf{j} + \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k} \\ &= \text{curl} \mathbf{F}. \end{aligned}$$

**Example 31** Find the curl of  $\mathbf{F} = (x^2 - y) \mathbf{i} + 4z \mathbf{j} + x^2 \mathbf{k}$ .

**Example 32** For a 1-form

$$\omega = Pdx + Qdy + Rdz$$

defined in a domain  $D$  in  $\mathbb{R}^3$  we obtain

$$d\omega = \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy \wedge dz + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz \wedge dx + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy$$

**Theorem 7.1** Let  $S$  be an oriented piecewise-smooth compact two dimensional surface with boundary  $\partial S$  embedded in a domain  $G \subset \mathbb{R}^3$ , in which a smooth 1-form  $\omega = Pdx + Qdy + Rdz$  is defined. Then the following relation holds:

$$\int_{\partial S} Pdx + Qdy + Rdz = \iint_S \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy \wedge dz + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz \wedge dx + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy$$

where the orientation of the boundary  $\partial S$  is chosen consistent with the orientation of the surface  $S$ .

In other notation, this means that

$$\int_S d\omega = \int_{\partial S} \omega.$$

Let us show the proof of the Stokes theorem. To avoid expressions that are really too cumbersome, we shall write out only the first, main part of its two

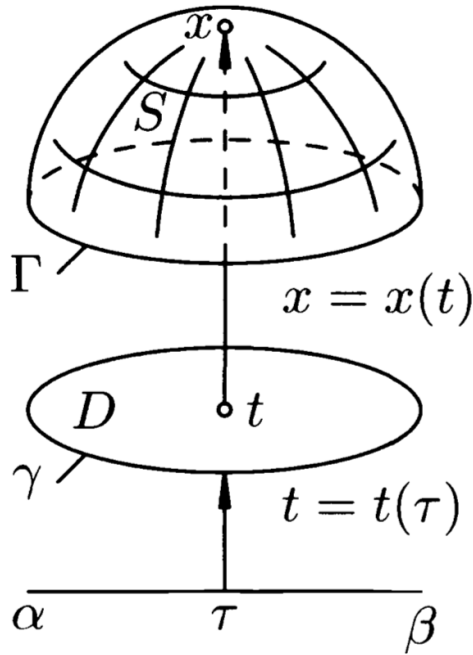


Figure 11: Parameters of a surface for Stokes.

expressions, and with some simplifications even in that. To be specific, let us introduce the notation  $x^1, x^2, x^3$  for the coordinates of a point  $x \in \mathbb{R}^3$  and verify only that

$$\int_{\partial S} P(x) dx^1 = \iint_S \frac{\partial P}{\partial x^2} dx^2 \wedge dx^1 + \frac{\partial P}{\partial x^3} dx^3 \wedge dx^1$$

For simplicity we shall assume that  $S$  can be obtained by a smooth mapping  $x = x(t)$  of domain  $D$  in the plane  $\mathbb{R}^2$  of the variables  $t^1, t^2$  and bounded by a smooth curve  $\gamma = \partial D$  parametrized via a mapping  $t = t(\tau)$  by the points of the closed interval  $\alpha < \tau < \beta$  (See Figure 11). Then the boundary  $\Gamma = \partial S$  of the surface  $S$  can be written as  $x = x(t(\tau))$ , where  $\tau$  ranges over the closed interval  $[\alpha, \beta]$ . Using the definition of the integral over a curve, Green's formula for a plane domain  $D$ , and the definition of the integral over a parametrized surface, we find

successively.

$$\begin{aligned}
\int_{\Gamma} P(x) dx^1 &= \int_{\alpha}^{\beta} P(x(t(\tau))) \left( \frac{\partial x^1}{\partial t^1} \frac{dt^1}{d\tau} + \frac{\partial x^1}{\partial t^2} \frac{dt^2}{d\tau} \right) d\tau \\
&= \int_{\gamma} P(x(t)) \left( \frac{\partial x^1}{\partial t^1} dt^1 + \frac{\partial x^1}{\partial t^2} dt^2 \right) \\
&= \iint_D \left[ \frac{\partial}{\partial t^1} \left( P \frac{\partial x^1}{\partial t^2} \right) - \frac{\partial}{\partial t^2} \left( P \frac{\partial x^1}{\partial t^1} \right) \right] dt^1 \wedge dt^2 \\
&= \iint_D \left( \frac{\partial P}{\partial t^1} \frac{\partial x^1}{\partial t^2} - \frac{\partial P}{\partial t^2} \frac{\partial x^1}{\partial t^1} \right) dt^1 \wedge dt^2 \\
&= \iint_D \sum_{i=1}^3 \left( \frac{\partial P}{\partial x^i} \frac{\partial x^i}{\partial t^1} \frac{\partial x^1}{\partial t^2} - \frac{\partial P}{\partial x^i} \frac{\partial x^i}{\partial t^2} \frac{\partial x^1}{\partial t^1} \right) dt^1 \wedge dt^2 \\
&= \iint_D \left( \frac{\partial P}{\partial x^2} \frac{\partial x^2}{\partial t^1} + \frac{\partial P}{\partial x^3} \frac{\partial x^3}{\partial t^1} \right) \frac{\partial x^1}{\partial t^2} \\
&\quad - \left( \frac{\partial P}{\partial x^2} \frac{\partial x^2}{\partial t^2} + \frac{\partial P}{\partial x^3} \frac{\partial x^3}{\partial t^2} \right) \frac{\partial x^1}{\partial t^1} dt^1 \wedge dt^2 \\
&= \iint_D \left( \frac{\partial P}{\partial x^2} \left| \frac{\partial(x^2, x^1)}{\partial(t^1, t^2)} \right| + \frac{\partial P}{\partial x^3} \left| \frac{\partial(x^3, x^1)}{\partial(t^1, t^2)} \right| \right) dt^1 \wedge dt^2 \\
&= \iint_S \left( \frac{\partial P}{\partial x^2} dx^2 \wedge dx^1 + \frac{\partial P}{\partial x^3} dx^3 \wedge dx^1 \right)
\end{aligned}$$

## 8 作业

1. 计算下列第一类曲线积分

(a)  $\int_L |y| ds$ , 其中 $L$ 为单位圆周 $x^2 + y^2 = 1$ .

(b)  $\int_L |x|^{1/3} ds$ , 其中 $L$ 为星形线 $x^{2/3} + y^{2/3} = a^{2/3}$ .

(c)  $\int_L |x| ds$ , 其中 $L$ 为双纽线 $(x^2 + y^2)^2 = x^2 - y^2$ .

(d)  $\int_L (x^2 + y^2 + z^2) ds$ , 其中 $L$ 为一段螺旋线 $x = a \cos t, y = a \sin t, z = bt, 0 \leq t \leq 2\pi$ .

(e)  $\int_L (xy + yz + zx) ds$ , 其中 $L$ 为球面 $x^2 + y^2 + z^2 = a^2$ 和平面 $x + y + z = 0$ 的交线.

2. 计算下列曲面的面积

(a) 圆柱面 $x^2 + y^2 = a^2$ 被两平面 $x + z = 0, x - z = 0 (x > 0, y > 0)$ 所截的部分。

(b) 抛物面 $x^2 + y^2 = 2az$ 包含在柱面 $(x^2 + y^2)^2 = 2a^2xy (a > 0)$ 内的那部分。

3. 计算下列第一类曲面积分:

(a)  $\iint_{\Sigma} (x + y + z) dS$ , 其中 $\Sigma$ 为左半球面 $x^2 + y^2 + z^2 = a^2, y \leq 0$ .

(b)  $\iint_{\Sigma} \left( \frac{x^2}{2} + \frac{y^2}{3} + \frac{z^2}{4} \right) dS$ , 其中 $\Sigma$ 为球面 $x^2 + y^2 + z^2 = a^2$ .

(c)  $\iint_{\Sigma} z dS$ , 其中 $\Sigma$ 螺旋面 $x = u \cos v, y = u \sin v, z = v, 0 \leq u \leq a, 0 \leq v \leq 2\pi$ .

4. 计算下列第二类曲线积分

(a)  $\int_L (x^2 + y^2) dx + (x^2 - y^2) dy$  其中 $L$ 是以 $A(1, 0), B(2, 0), C(2, 1), D(1, 1)$ 为顶点的正方向, 方向为逆时针方向。

(b)  $\int_L (x^2 - 2xy) dx + (y^2 - 2xy) dy$  其中 $L$ 是抛物线上的一段:  $y = x^2, -1 \leq x \leq 1$ , 方向由 $(-1, 1)$ 到 $(1, 1)$ 。

(c)  $\int_L \frac{(x+y)dx - (x-y)dy}{x^2 + y^2}$  其中  $L$  是圆周  $x^2 + y^2 = a^2$ , 方向为逆时针方向。

(d)  $\int_L xdx + ydy + (x+y-1)dz$  其中  $L$  是  $(1, 1, 1)$  到  $(2, 3, 4)$  的直线段。

(e)  $\int_L ydx + zdy + xdz$  其中  $L$  是曲线  $\begin{cases} x^2 + y^2 + z^2 = 2az \\ x + z = a (a > 0) \end{cases}$ , 从  $z$  轴的正向看去, 曲线  $L$  方向为逆时针方向。

5. 证明不等式:

$$\left| \int_L P(x, y) dx + Q(x, y) dy \right| \leq MC$$

其中  $C$  是曲线  $L$  的弧长,  $M = \max \left\{ \sqrt{P^2(x, y) + Q^2(x, y) + R^2(x, y)} \mid (x, y) \in L \right\}$

6. 计算下列第二类曲面积分

(a)  $\int_{\Sigma} (x+y)dydz + (y+z)dzdx + (z+x)dxdy$  其中  $\Sigma$  是中心在原点, 边长为  $2h$  的立方体  $[-h, h] \times [-h, h] \times [-h, h]$ , 方向取外向。

(b)  $\int_{\Sigma} yzdzdx$  其中  $\Sigma$  是椭球线  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$  的上半部分, 方向取上侧。

(c)  $\int_{\Sigma} [f(x, y, z) + x] dydz + [2f(x, y, z) + y] dzdx + [f(x, y, z) + z] dxdy$  其中  $f(x, y, z)$  为连续函数,  $\Sigma$  是平面  $x - y + z = 1$  在第四象限部分, 方向取上侧。

7. 利用Green公式计算下列积分

(a)  $\int_L (x+y)^2 dx - (x^2 + y^2) dy$  其中  $L$  是以  $A(1, 1), B(3, 2), C(2, 5)$  为顶点的三角形的边界, 方向为逆时针方向。

(b)  $\int_L xy^2 dx - x^2y dy$  其中  $L$  是圆周  $x^2 + y^2 = a^2$  方向为逆时针方向。

(c)  $\int_L (x^2y \cos x + 2xy \sin x - y^2e^x) dx + (x^2 \sin x - 2ye^x) dy$  其中  $L$  是星形线  $x^{2/3} + y^{2/3} = a^{2/3}$ , 方向为逆时针方向。

(d)  $\int_L \frac{e^x [(x \sin y - y \cos y) dx + (x \cos y - y \sin y) dy]}{x^2 + y^2}$  其中  $L$  是包含原点的简单闭曲线, 方向为逆时针方向。

8. 利用曲线积分, 求下列曲线所围成图形的面积

(a) 星形线  $x^{2/3} + y^{2/3} = a^{2/3}$ .



(b) 抛物线 $(x + y)^2 = ax(a > 0)$ 与 $x$ 轴.

9. 先证明曲线积分与路径无关, 在计算积分值

(a)  $\int_{(0,0)}^{(1,1)} (x - y)(dx - dy)$ .

(b)  $\int_{(1,0)}^{(6,8)} \frac{x dx + y dy}{\sqrt{x^2 + y^2}}$ .

10. 证明:  $(2x \cos y + y^2 \cos x)dx + (2y \sin x - x^2 \sin y)dy$ 在整个平面上是某个函数的全微分, 并找出它的一个原函数.

11. 设 $Q(x, y)$ 在 $xy$ 平面山具有连续的偏导数, 曲线积分 $\int_L 2xy dx + Q(x, y)dy$ 与路径无关, 并且对于任意 $t$ 恒有,

$$\int_{(0,0)}^{(t,1)} 2xy dx + Q(x, y)dy = \int_{(0,0)}^{(1,t)} 2xy dx + Q(x, y)dy.$$

求 $Q(x, y)$ .

12. 利用Gauss公式计算下列积分

(a)  $\iint_{\Sigma} x^2 dy dz + y^2 dz dx + z^2 dx dy$ , 其中 $\Sigma$ 为立方体 $0 \leq x, y, z \leq a$ 的表面, 方向为外侧。

(b)  $\iint_{\Sigma} (x - y + z) dy dz + (y - z + x) dz dx + (z - x + y) dx dy$ , 其中 $\Sigma$ 为闭曲面 $|x - y + z| + |y - z + x| + |z - x + y| = 1$ , 方向为外侧。

(c)  $\iint_{\Sigma} (x^2 \cos \alpha + y^2 \cos \beta + z^2 \cos \gamma) dS$ , 其中 $\Sigma$ 为锥面 $z^2 = x^2 + y^2$ 介于平面 $z = 0$ 与 $z = h(h > 0)$ 之间的部分, 方向取下侧。

(d)  $\iint_{\Sigma} \frac{x dy dz + y dz dx + z dx dy}{(x^2 + y^2 + z^2)^{3/2}}$ , 其中 $\Sigma$ 为:

i. 椭球面 $x^2 + 2y^2 + 3z^2 = 1$ , 方向为外侧;

ii. 抛物面 $1 - \frac{z}{5} = \frac{(x - 2)^2}{16} + \frac{(y - 1)^2}{9} (z \geq 0)$ , 方向取上侧。