The Differential of a Function of Several Variables

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April 28, 2019

1 Double Integral

1.1 Introduction

Volume Problem: Find the volume V of the solid G enclosed between the surface $z = f(x, y)$ and the region R in the $xy-$plane where $f(x, y)$ is continuous and non-negative on R.

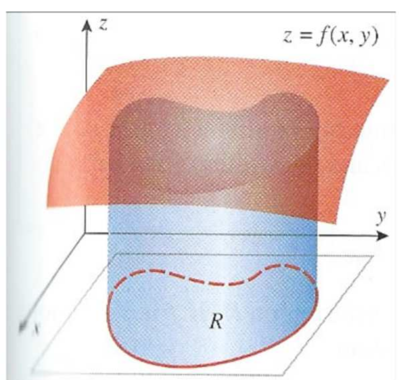


Figure 1: Volume.

Mass Problem: Find the mass M of a lamina (a region R in the $xy-$plane) whose density is a continuous nonnegative function $\rho(x, y)$

Let us consider the Volume Problem,

1. Divide the rectangle enclosing R into subrectangles, and exclude all those rectangles that contain points outside R. Let n be the number of all the rectangles inside R, and let $\Delta A_k = \Delta x_k \Delta y_k$ be the area of the $k-$th subrectangle.

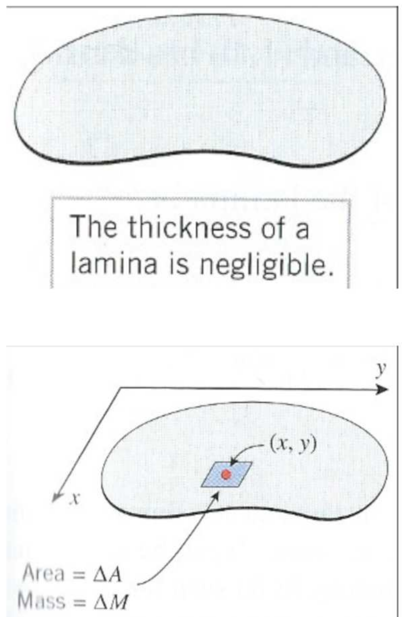


Figure 2: Mass.

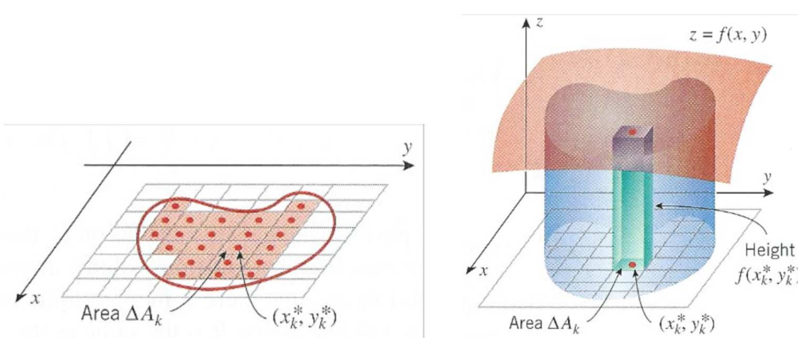


Figure 3: Volume.

2. Choose any point (ξ_k, η_k) in the $k-$th subrectangle. The volume of a rectangular parallelepiped with base area ΔA_k and height $f(\xi_k, \eta_k)$ is $\Delta V_k = f(\xi_k, \eta_k)\Delta A_k$, thus,

$$V \approx \sum_{k=1}^{n} \Delta V_k = \sum_{k=1}^{n} f(\xi_k, \eta_k)\Delta A_k = \sum_{k=1}^{n} f(\xi_k, \eta_k)\Delta x_k \Delta y_k \tag{1}$$

This sum is called the *Rimann sum*.

3. Take the sides of all the subrectangles to 0, and get

$$V = \lim_{\lambda(P)\to 0} \sum_{k=1}^{n} f(\xi_k, \eta_k)\Delta A_k = \iint_R f(x, y)\, \mathrm{d}A \tag{2}$$

The last term is the notation for the limit of the Riemann sum, and it is called the *double integral* of $f(x, y)$ over R.

1.2 The Darboux Criterion

Let us consider another criterion for Riemann integrability of a function, which is applicable only to real-valued function. Lower and Upper Darboux Sums. Let f be a real-valued function on the interval I and $P = I_i$ a partition of the interval I. We set

$$m_i = \inf_{x\in I_i} f(x), M_i = \sup_{x\in I_i} f(x). \tag{3}$$

The quantities

$$s(f, P) = \sum_i m_i|I_i|, S(f, P) = \sum_i M_i|I_i| \tag{4}$$

are called the *lower* and *upper sums* of the function f over the interval I corresponding to the partion P of the interval.

The following relations hold between the Darboux sums a a function $f : I \to \mathbb{R}$:

1. $s(f, P) = \inf_\xi \sigma(f, P, \xi) \leq \sigma(f, P, \xi) \leq \sup_\xi \sigma(f, P, \xi) = S(f, P)$
2. If the partition P' of the interval I is obtained by refining intervals of the partion P, then $s(f, P) \leq s(f, P') \leq S(f, P') \leq S(f, P)$
3. The inequality $s(f, P_1) \leq S(f, P_2)$ holds for any pair of partition P_1, P_2 of the interval I.

1.3 Lower and Upper Integrals

The lower and upper Darboux integrals of the function $f : I \to \mathbb{R}$ over the interval I are respectively

$$\underline{I} = \sup_{P} s(f, P), \overline{I} = \inf_{P} S(f, P) \tag{5}$$

where the supremum and infimum are taken over all partitions P of the interval I.

Theorem 1.1. *For any bounded function:*

$$f : I \to \mathbb{R}, \lim_{\lambda(P)\to 0} s(f, P) = \underline{I}, \lim_{\lambda(P)\to 0} S(f, P) = \overline{I}$$

1.4 The Darboux Criterion for Integrability of a Real-valued Function

Theorem 1.2 ((The Darboux Criterion)**.** *A real valued function $f : I \to \mathbb{R}$ defined on an interval $I \subset \mathbb{R}^{\ltimes}$ is integrable over that interval if and only if it is bounded on I and its upper and lower Darboux integrals are equal.*

$$f \in \mathcal{R}(I) \equiv f \textit{ is bounded on } I, \textit{and} \underline{I} = \overline{I}.$$

1.5 Double Integral Over Non-rectangular Regions

$$\int_a^b \int_c^d f(x, y)\, \mathrm{d}y\mathrm{d}x = \int_a^b \left[\int_c^d f(x, y)\right] \mathrm{d}x \tag{6}$$

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, \mathrm{d}y\mathrm{d}x = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x, y)\mathrm{d}y\right] \mathrm{d}x \tag{7}$$

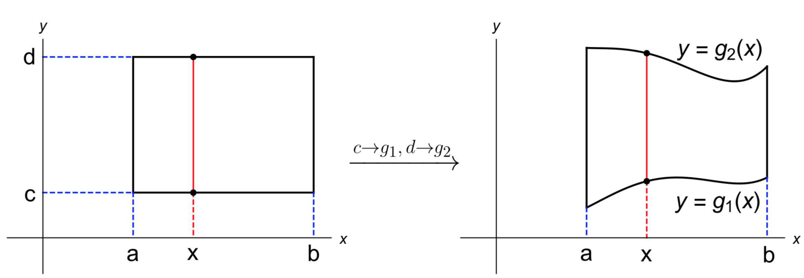


Figure 4: Double Integral.

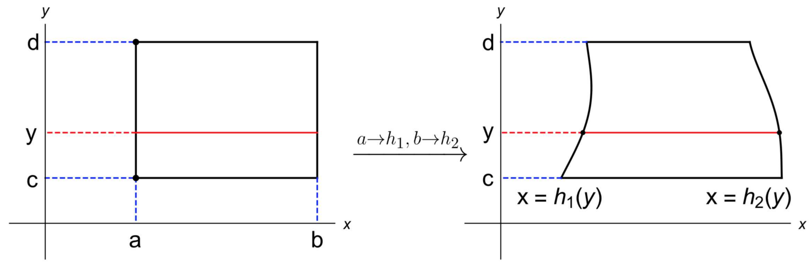


Figure 5: Double Integral.

$$\int_a^b \int_{h_1(y)}^{h_2(y)} f(x,y)\,\mathrm{d}x\mathrm{d}y = \int_a^b \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\mathrm{d}x\right] \mathrm{d}y \tag{8}$$

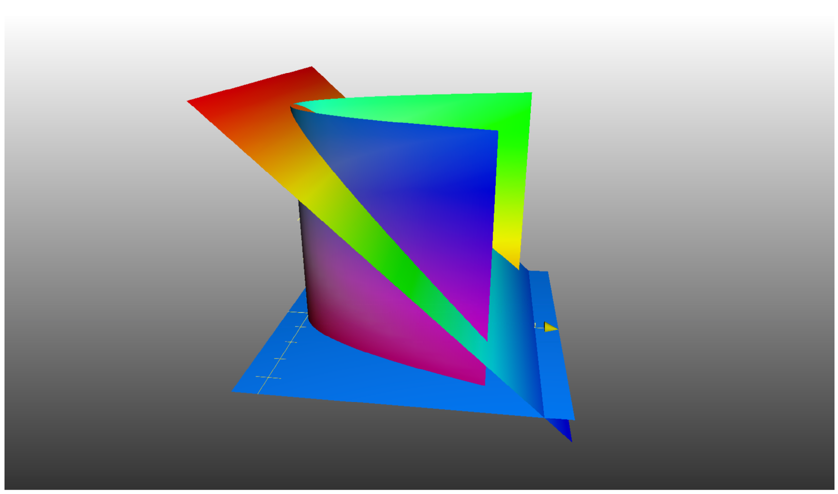
Examples 1. *1.* $\int_0^1 \int_1^2 \sin(2x-3y)\mathrm{d}x\mathrm{d}y$

2. $\int_1^2 \int_0^1 \sin(2x-3y)\mathrm{d}y\mathrm{d}x$

Theorem 1.3. *Let D be the rectangle $[a,b]\times[c,d]$. If $f(x,y)$ is continuous on D then*

$$\iint f(x,y)\,\mathrm{d}\sigma = \int_a^b \int_c^d f(x,y)\,\mathrm{d}y\mathrm{d}x = \int_c^d \int_a^b f(x,y)\,\mathrm{d}x\mathrm{d}y$$

Examples 2. *Find the volume V of the solid enclosed by the surfaces $z = 0, y^2 = x, x+z = 1$*

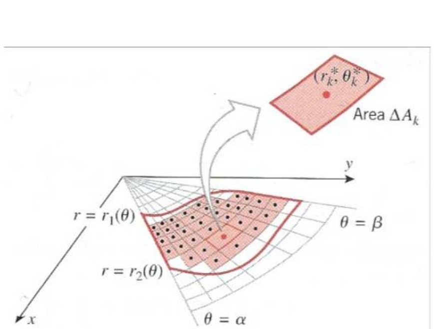


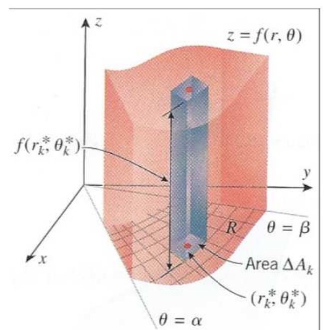
Examples 3. *Reverse the order of the integrals below:*

1. $\int_a^b \mathrm{d}x \int_a^x f(x,y)\mathrm{d}y$
2. $\int_0^1 \mathrm{d}y \int_0^{2y} f(x,y)\mathrm{d}x + \int_1^3 \mathrm{d}y \int_0^{3-y} f(x,y)\mathrm{d}x$

1.6 Double integrals in polar coordinates

Let us consider the volume problem in polar coordinates





$$\iint_D f(x,y)\,\mathrm{d}\sigma = \int_\alpha^\beta \mathrm{d}\theta \int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta) r\,\mathrm{d}r$$

Examples 4.

$$\iint_D \frac{\mathrm{d}\sigma}{\sqrt{1-x^2-y^2}}$$

$D : \{(x,y) |\, x^2+y^2 \leq 1\}$

Examples 5. *Compute the volume of the Figure(1.6), which the surfaces are*

$$x^2+y^2+z^2 \leq R^2, x^2+y^2 = Rx$$

Examples 6.

$$\iint_D e^{-(x^2+y^2)}\,\mathrm{d}\sigma.$$

$D : x^2+y^2 \leq R^2$

Examples 7. *Compute the volume*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \leq 1$$

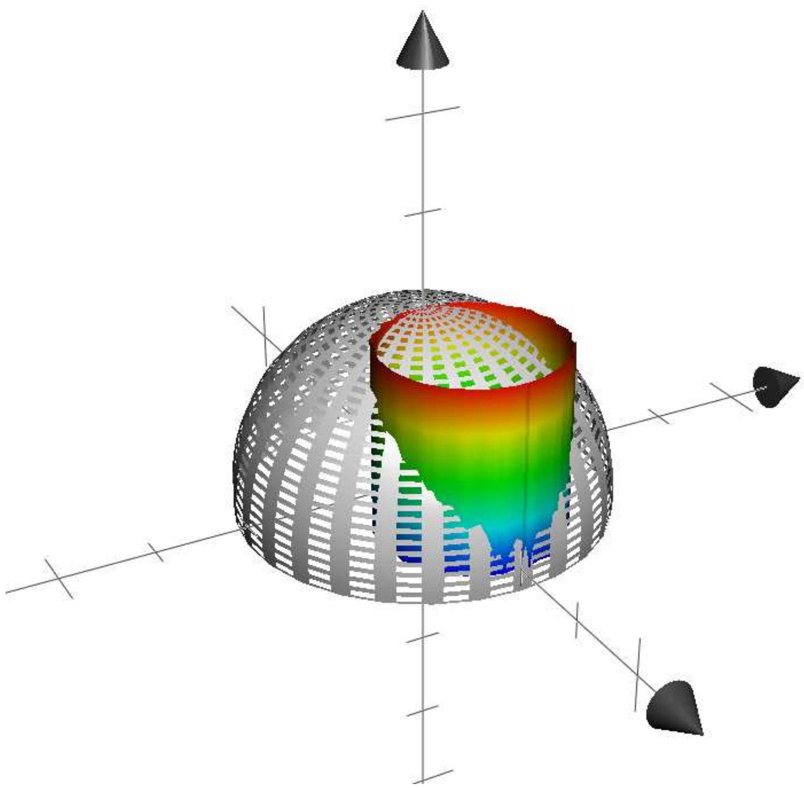


Figure 6: Volume.

Examples 8.

$$\iint_D e^{\frac{x-y}{x+y}}\,\mathrm{d}\sigma.$$

$D : x + y \le 1, x \ge 0, y \ge 0.$

Examples 9.

$$\int_{-\infty}^{+\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}$$

1.7 Triple Integrals

Find the mass M of a solid G whose density (the mass per unit volume) is a continuous nonnegative function $\rho(x, y, z)$.

1.7.1 Evaluating triple integrals over rectangular boxex

$$\iiint_G f(x,y,z)\,\mathrm{d}v = \int_a^b \mathrm{d}x \int_c^d \mathrm{d}y \int_k^l f(x,y,z)\,\mathrm{d}z$$

$$\iiint_G f(x,y,z)\,\mathrm{d}v = \iint_D \mathrm{d}\sigma \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,\mathrm{d}z$$

$$\iiint_G f(x,y,z)\,\mathrm{d}v = \int_a^b \mathrm{d}x \iint_D f(x,y,z)\,\mathrm{d}\sigma$$

Examples 10.

$$\iiint_V \frac{\mathrm{d}x\mathrm{d}y\mathrm{d}z}{x^2+y^2}$$

$V : x = 1, x = 2, z = 0, y = x, z = y$

Examples 11.

$$\iiint_V (x^2 + y^2 + z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$V : z = \sqrt{x^2 + y^2}, z = 1$

Examples 12.

$$\iiint_V \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)\mathrm{d}v$$

$V : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1$

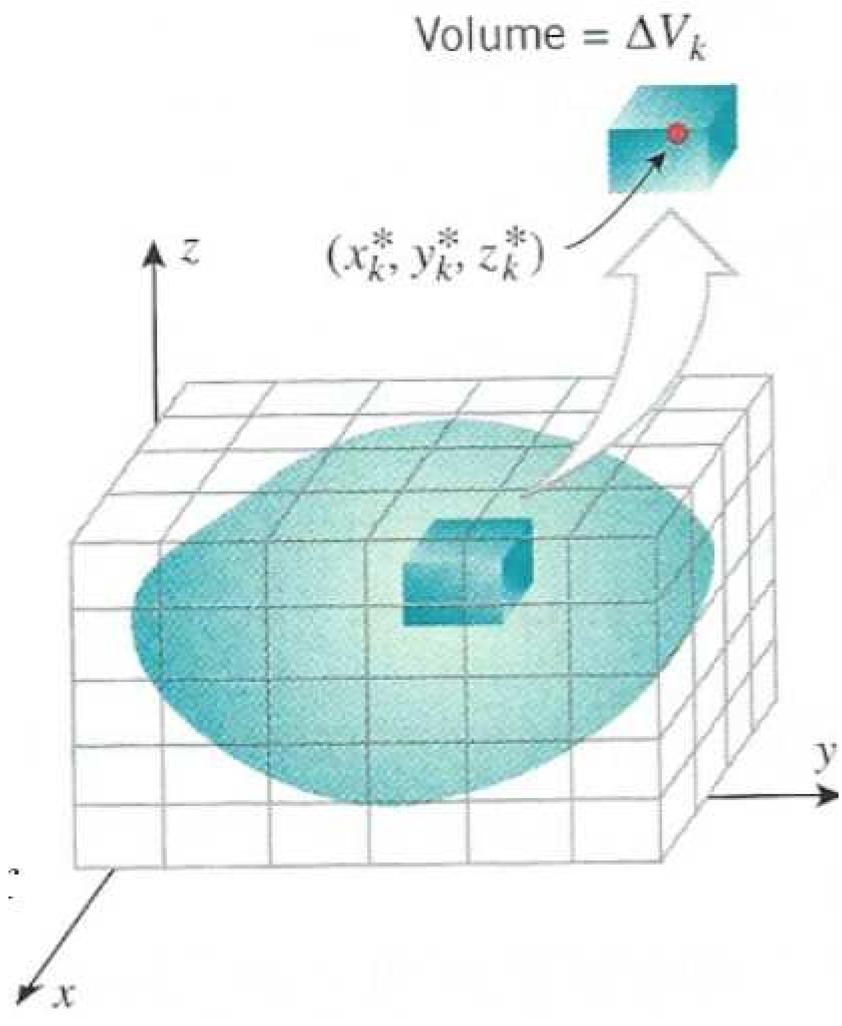


Figure 7: Mass.

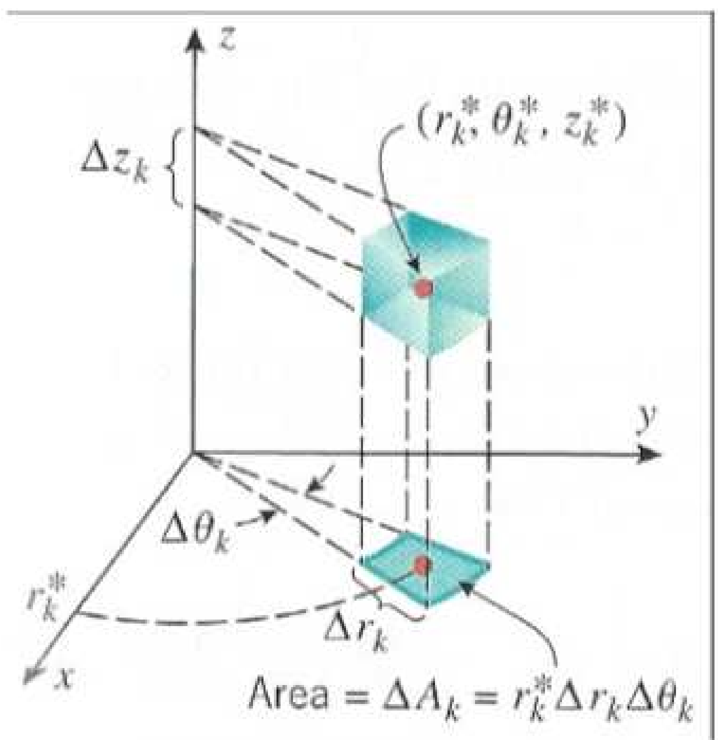


Figure 8: Cylindrical.

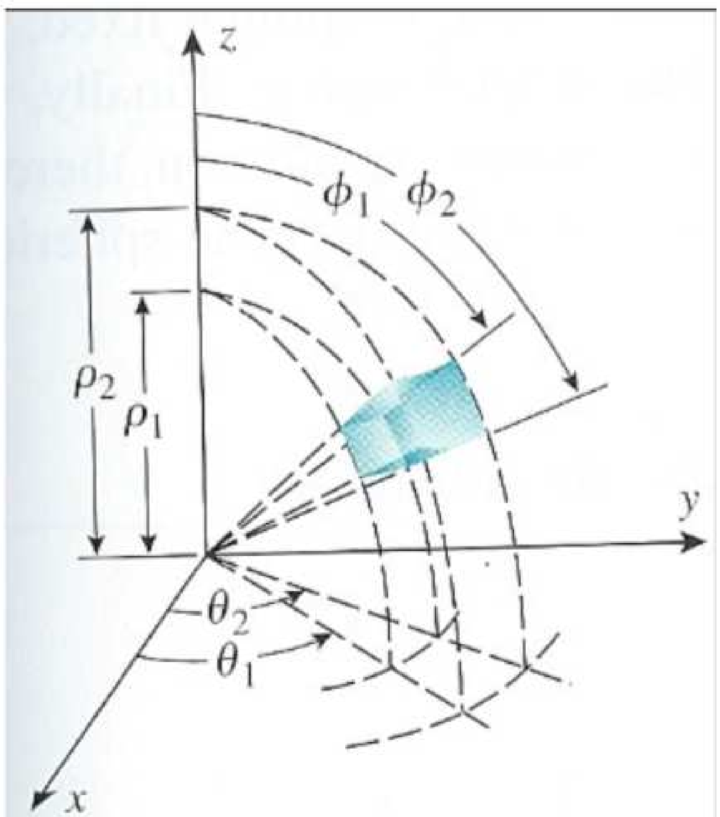


Figure 9: Spherical.

1.7.2 Triple integrals in cylindrical and spherical coordinates

$$\iiint_V f(x,y,z)\,\mathrm{d}v = \iiint_{V'} f(r\cos\theta, r\sin\theta, z) r\,\mathrm{d}r\mathrm{d}\theta\mathrm{d}z.$$

$$\iiint_V f(x,y,z)\,\mathrm{d}v = \iiint_{V'} f(r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi) r^2\sin\phi\,\mathrm{d}r\mathrm{d}\phi\mathrm{d}\theta.$$

Examples 13.

$$\iiint_\omega \left(x^2+y^2\right)\,\mathrm{d}v,$$

ω *is the volume bounded by surfaces* $z = x^2+y^2, z = h$

Examples 14.

$$\iiint_\omega z e^{-(x^2+y^2+z^2)}\,\mathrm{d}v,$$

ω *is the volume bounded by surfaces* $z = \sqrt{x^2+y^2}, x^2+y^2+z^2 = 1$

Examples 15. *Compute the volume of the* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1$

Examples 16. *Compute the volume of* $x_1^2+x_2^2+\cdots+x_n^2 \le R^2$

Examples 17. *Compute the area inside* $(x-y)^2+x^2 = a^2 (a>0)$.

Examples 18. *Compute the area bounded by* $xy = p, xy = q, y = ax, y = bx$.

Examples 19. *Compute the volume bounded by* $(x^2+y^2)^2+z^4 = y$.

Examples 20. *Compute the volume bounded by* $x_1^2+x_2^2+\cdots+x_n^2 \le R^2$.

2 作業

1. 根據二重積分的性質，比較下列積分的大小

   (a) $\iint_D (x+y)^2\,\mathrm{d}x\mathrm{d}y, \iint_D (x+y)^3\,\mathrm{d}x\mathrm{d}y$，其中$D$為$x$軸，$y$軸與直線$x+y=1$ 所圍成的區域。

   (b) $\iint_D \ln(x+y)\,\mathrm{d}x\mathrm{d}y, \iint_D [\ln(x+y)]^2\,\mathrm{d}x\mathrm{d}y$，其中$D=[3,5]\times[0,1]$。

2. 計算下列重積分

   (a) $\iint_D x^3+3x^2y+y^3\,\mathrm{d}x\mathrm{d}y$，其中$D=[0,1]\times[0,1]$。

   (b) $\iint_D xye^{x^2+y^2}\,\mathrm{d}x\mathrm{d}y$，其中$D=[a,b]\times[c,d]$。

   (c) $\iint_D xy^2\,\mathrm{d}x\mathrm{d}y$，其中$D$由拋物線$y^2=2px$ 和直線$x=\frac{p}{2}(p>0)$所圍成的區域。

   (d) $\iint_D e^{x+y}\,\mathrm{d}x\mathrm{d}y$，其中$D$為$\{(x,y)|\,|x|+|y|\leq 1\}$所圍成的區域。

   (e) $\iint_D x\,\mathrm{d}x\mathrm{d}y$，其中$D$為拱線的一拱$x=a(t-\sin t), y=a(1-\cos t)(0\leq t\leq 2\pi)$所圍成的區域。

3. 採用極坐標變換計算下列積分

   (a) $\iint_D \sin\sqrt{x^2+y^2}\,\mathrm{d}\sigma, D=\left\{(x,y)|\,\pi^2\leq x^2+y^2\leq 4\pi^2\right\}$

   (b) $\iint_D (x+y)\,\mathrm{d}\sigma, D=\left\{(x,y)|\,x^2+y^2\leq x+y\right\}$

   (c) $\iint_D |xy|\,\mathrm{d}\sigma, D=\left\{(x,y)|\,x^2+y^2\leq a^2\right\}$

   (d) $\iint_D f'(x^2+y^2)\,\mathrm{d}\sigma, D=\left\{(x,y)|\,x^2+y^2\leq R^2\right\}$

4. 做適當的座標變換計算下列積分

   (a) $\iint_D (x+y)\sin(x-y)\,\mathrm{d}\sigma, D=\{(x,y)|\,0\leq x+y\leq\pi, 0\leq x-y\leq\pi\}$

   (b) $\iint_D e^{\frac{x}{x+y}}\,\mathrm{d}\sigma, D=\{(x,y)|\,x+y\leq 1, x\geq 0, y\geq 0\}$

5. 計算下列幾何體的體積

   (a) V是由$z=x^2+y^2, z=x+y$所圍成的立體。

(b) V是由$z^2 = \dfrac{x^2}{4} + \dfrac{y^2}{9}$和$2z^2 = \dfrac{x^2}{4} + \dfrac{y^2}{9}$所圍成的立體。

6. 選取適當的座標變換計算下列積分

   (a) $\iiint_\Omega (x^2 + y^2 + z^2)\,\mathrm{d}v$ 其中Ω為球$x^2 + y^2 + z^2 \leq 1$

   (b) $\iiint_\Omega \sqrt{1 - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2}}\,\mathrm{d}v$ 其中Ω為球$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} \leq 1$

   (c) $\iiint_\Omega \dfrac{z\ln(1 + x^2 + y^2 + z^2)}{1 + x^2 + y^2 + z^2}\,\mathrm{d}v$ 其中Ω為球$x^2 + y^2 + z^2 \leq 1, z \geq 0$

   (d) $\iiint_\Omega (x + y - z)(x - y + z)(y + z - x)\,\mathrm{d}v$ 其中Ω為$0 \leq x + y - z \leq 1, 0 \leq x - y + z \leq 1, 0 \leq y + z - x \leq 1$

7. 計算下列積分

   (a) $\int_\Omega \sqrt{x_1 + x_2 + \cdots + x_n}\,\mathrm{d}v$ 其中Ω為$x_1 + x_2 + \cdots + x_n \leq 1, x_i \geq 0 (i = 1, 2, \cdots, n)$

   (b) $\int_\Omega \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}\,\mathrm{d}v$ 其中Ω為$x_1^2 + x_2^2 + \cdots + x_n^2 \leq 1$