Derivative

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1 Differentiable Functions

Definition 1.1. A function $f : E \to \mathbb{R}$ defined on a set E is differentiable at a point $a \in E$ that is a limit point of E if there exists a linear function $A(x-a)$ of the increment $x-a$ of the argument such that $f(x)-f(a)$ can be represented as

$$f(x) - f(a) = A(x-a) + \circ(x-a) \text{ as } x \to a, a \in E \tag{1}$$

In other words, a function is differentiable at a point a if the change in its values in a neighborhood of the point in question is linear up to a correction that is infinitesimal compared with the magnitude of the displacement $x-a$ for the point a.

Definition 1.2. The linear function $A(x-a)$ in Eq. 1 is called the differential of the function f at a.

The number A is unambiguously determined due to the uniqueness of the limit.

Definition 1.3. The number

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{2}$$

is called the derivative of the function f at a.

Graphically, this definition says that the derivative of f at a is the slope of the tangent line to $y = f(x)$ at a, which is the limit as $x \to a$ of the slopes of the lines through $(x, f(x))$ and $(a, f(a))$.

We can also write

$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}$$

Definition 1.4. A function $f : E \to \mathbb{R}$ defined on a set $E \subset \mathbb{R}$ is differentiable at a point $x \in E$ that is a limit point of E if

$$f(x+h) - f(x) = A(x)h + \alpha(x;h) \tag{3}$$

where $h \to A(x)h$ is a linear function in h and $\alpha(x;h) = \circ(h)$ as $h \to 0, x+h \in E$.

Definition 1.5. The function $h \to A(x)h$ of Definition 3, which is linear in h, is called the differential of the function $f : E \to \mathbb{R}$ at the point $x \in E$ and is denoted as $df(x)$ or $Df(x)$.

Thus, $df(x)(h) = A(x)h$.

From definitions 2 and 3 we have

$$\Delta f(x;h) - df(x)(h) = \alpha(x;h)$$

1.1 Some Examples

Examples 1. Let $f(x) = \sin x$. We shall show that $f'(x) = \cos x$.

Examples 2. We shall show that $\cos'(x) = -\sin x$.

Examples 3. If $f(t) = r \sin \omega t$, then $f'(t) = r\omega \cos \omega t$. If $f(t) = r \cos \omega t$, then $f'(t) = -r\omega \sin \omega t$.

Examples 4. The instantaneous velocity and instantaneous acceleration of a point mass. Suppose a point mass is moving in a plane and that in some given coordinate system its motion is described by differentiable function of time

$$x = x(t), y = y(t)$$

In particular, this motion is written as in the form

$$r(t) = (r\cos(\omega t + \alpha), r\sin(\omega t + \alpha))$$

Examples 5. The optic property of a parabolic mirror. Let us consider the parabola $y = \frac{1}{2p}x^2 (p > 0)$, and construct the tangent to it at the point $(x_0, y_0) = (x_0, \frac{1}{2p}x_0^2)$.

Examples 6.

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Examples 7. We shall show that

$$e^{x+h} - e^x = e^x h + \circ(h)$$

as $h \to 0$.

Examples 8. If $a > 0$, then $a^{x+h} - a^x = a^h(\ln a)h + \circ(h)$ as $h \to 0$.

2 The Basic Rules of Differentiation

2.1 Differentiation and the Arithmetic Operations

Theorem 2.1. If function $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are differentiable at a point $x \in X$, then a) their sum is differentiable at x, and

$$(f + g)'(x) = (f' + g')(x),$$

b) their product is differentiable at x, and

$$(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x),$$

c) their quotient is differentiable at x if $g(x) \neq 0$, and

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$

Corollary 2.2. The derivative of a linear combination of differentiable functions equals the same linear combination of the derivatives of these functions.

Corollary 2.3. If the functions $f_1, \cdots, f_n$ are differentiable at x, then

$$\begin{aligned}(f_1 f_2 \cdots f_n)'(x) = {} & f_1' f_2 \cdots f_n \\ & + f_1 f_2' \cdots f_n + \cdots + f_1 f_2 \cdots f_n'\end{aligned}$$

Corollary 2.4. It follows from the relation between the derivative and the differential that we have:

$$\begin{aligned}&a) d(f + g)(x) = df(x) + dg(x), \\ &b) d(f \cdot g)(x) = g(x)df(x) + f(x)dg(x), \\ &c) d\left(\frac{f}{g}\right)(x) = \frac{g(x)df(x) - f(x)dg(x)}{g^2(x)}.\end{aligned}$$

Examples 9. Find the derivative of $\tan x$ and $\cot x$.

2.2 Differentiation of a Composite Function (chain rule)

Theorem 2.5. If the function: $f : X \to Y \subset \mathbb{R}$ is differentiable at a point $x \in X$ and the function $g : Y \to \mathbb{R}$ is differentiable at the point $y = f(x) \in Y$, then the composite function $g \circ f : X \to \mathbb{R}$ is differentiable at x, and the differential $\mathrm{d(g \circ f)(x)} : \mathrm{T}\mathbb{R} \to \mathrm{T}\mathbb{R}\mathrm{g(f(x))}$ of their composition equals the composition $\mathrm{dg(y)} \circ \mathrm{df(x)}$ of their differentials.

Proof. The conditions for differentiability of the function f and g have the form.

$$f(x+h)-f(x)=f'(x)h+o(h), h\to 0, x+h\in X$$

$$g(y+t)-g(y)=g'(y)t+o(t), t\to 0, y+t\in Y$$

We remark that in the second equality here the function $o(t)$ can be considered to be defined for $t=0$, and in the representation $o(t)=\gamma(t)t$, where $\gamma(t)\to 0$ as $t\to 0, y+t\in Y$. Setting $f(x)=y$ and $f(x+h)=y+t$, by the differentiability of f at the point x we conclude that $t\to 0$ as $h\to 0$ We now have

$$\gamma(f(x+h)-f(x))=\alpha(h)\to 0$$

as $h\to 0, x+h\in X$. and thus if $t=f(x+h)-f(x)$, then,

$$\begin{aligned} o(t) &= \gamma(f(x+h)-f(x))(f(x+h)-f(x)) \\ &= \alpha(h)(f'(x)h+o(h)) = \alpha(h)f'(x)h+\alpha(h)o(h) \\ &= o(h)+o(h)=o(h) \end{aligned}$$

$$\begin{aligned} (g\circ f)(x+h)-(g\circ f)(x) &= g(f(x+h))-g(f(x)) \\ &= g(y+t)-g(y)=g'(y)t+o(t) \\ &= g'(f(x))(f(x+h)-f(x))+o(f(x+h)-f(x)) \\ &= g'(f(x))(f'(x)h)+g'(f(x))(o(h))+o(f(x+h)-f(x)) \\ o(f(x+h)-f(x)) &= o(h) \end{aligned}$$

□

Corollary 2.6. The derivative $(g\circ f)'(x)$ of the composition of differentiable real-valued functions equals the product $g'(f(x))\cdot f'(x)$ of the derivatives of these functions computed at the corresponding points.

$$\frac{\Delta z}{\Delta x}=\frac{\Delta z}{\Delta y}\cdot\frac{\Delta y}{\Delta x}$$

Examples 10. Let us show that for $\alpha\in\mathbb{R}$ we have $\frac{\mathrm{d}x^\alpha}{\mathrm{d}x}=\alpha x^{\alpha-1}$ in the domain $x>0$, that is, $\mathrm{d}x^\alpha=\alpha x^{\alpha-1}\mathrm{d}x$

Examples 11. The derivative of the logarithm of the absolute value of a differentiable function is often called its logarithmic derivative.

$$\mathrm{d}\left(\ln|f|\right)(x)=\frac{f'(x)}{f(x)}\mathrm{d}x=\frac{\mathrm{d}f(x)}{f(x)}.$$

Examples 12. The absolute and relative errors in the value of a differentiable function caused by errors in the data for the argument.

$$f(x+h) - f(x) = f'(x)h + \alpha(x;h),$$

$$\frac{|f'(x)h|}{|f(x)|} = \frac{|\mathrm{d}f(x)h|}{|f(x)|}$$

2.3 Differentiation of an Inverse Function

Theorem 2.7. Let the function $f : X \to Y$ and $f^{-1} : Y \to X$ be mutually inverse and continuous at points x_0 and $f(x_0) = y_0 \in Y$ respectively. If f is differentiable at x_0 and $f'(x_0) \neq 0$, then f^{-1} is also differentiable at the point y_0, and

$$\left(f^{-1}\right)'(y_0) = \left(f'(x_0)\right)^{-1}.$$

Remark. If we knew in advance that the function f^{-1} was differentiable at y_0, we would find immediately by the identity $\left(f^{-1} \circ f\right)(x) = x$ and the theorem on differentiation of a composite function that $\left(f^{-1}\right)' \cdot f'(x_0) = 1$.

Remark. The condition $f'(x_0) \neq 0$ is obviously equivalent to the statement that the mapping $h \to f'(x_0)h$ realized by the differential $\mathrm{d}f(x_0) : T\mathbb{R}(x_0) \to T\mathbb{R}(y_0)$ is invertible mapping $[\mathrm{d}f(x_0)]^{-1} : T\mathbb{R}(y_0) \to T\mathbb{R}(x_0)$ given by the formula $\tau \to \left(f'(x_0)\right)^{-1}\tau$.

Examples 13. We shall show that $\arcsin' y = \frac{1}{1-y^2}$ for $|y| < 1$.

Examples 14. $arccot'y = -\dfrac{1}{1+y^2}, \arctan' y = \dfrac{1}{1+y^2}$

Examples 15. The hyperbolic and inverse hyperbolic functions and their derivatives. The function

$$\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$$
$$\cosh x = \frac{1}{2}\left(e^x - e^{-x}\right)$$

are respectively the hyperbolic sine and hyperbolic cosine of x. These functions, which for the time being have been introduced purely formally, arise just as naturally in many problems as the circular functions $\sin x$ and $\cos x$.

We remark that

$$\sinh(-x) = -\sinh x$$
$$\cosh(-x) = \cosh x$$

Moreover, the following basic identity is obvious

$$\cosh^2 x - \sinh^2 x = 1$$

The graphs of the functions $y = \sinh x$ and $y = \cosh x$ are shown in Fig 2. The inverse of the hyperbolic sine is

$$x = \ln(y + \sqrt{1 + y^2})$$

Thus,

$$\sinh^{-1} y = \ln(y + \sqrt{1 + y^2})$$

Similarly, using the monotonicity of the function $y = \cosh x$ on its definition,

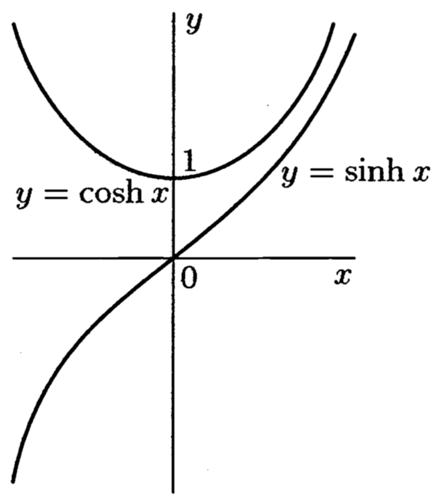

Figure 1: Hyperbolic functions.

we have

$$\cosh_{-}^{-1}(y) = \ln\left(y - \sqrt{y^2 - 1}\right)$$
$$\cosh_{+}^{-1}(y) = \ln\left(y + \sqrt{y^2 - 1}\right)$$

From the definitions given above, we find

$$\sinh' x = \cosh x,$$
$$\cosh' x = \sinh x,$$

and by the theorem on the derivative of an inverse function, we find

$$\left(\sinh^{-1} y\right)' = \frac{1}{\sinh' x} = \frac{1}{\cosh' x} = \frac{1}{\sqrt{1+y^2}}$$

$$\left(\cosh_{-}^{-1} y\right)' = \frac{1}{\cosh' x} = \frac{1}{\cosh' x} = \frac{1}{-\sqrt{\cosh^2 x - 1}} = -\frac{1}{\sqrt{y^2-1}}, y > 1$$

$$\left(\cosh_{+}^{-1} y\right)' = \frac{1}{\cosh' x} = \frac{1}{-\sqrt{\cosh^2 x - 1}} = \frac{1}{\sqrt{y^2-1}}, y > 1$$

Like $\tan x$ and $\cot x$ one can consider the functions

$$\tanh x = \frac{\sinh x}{\cosh x}, and \coth x = \frac{\cosh x}{\sinh x}$$

called the hyperbolic tangent and hyperbolic cotangent respectively, and also the functions inverse to them, the area tangent

$$\tanh^{-1} y = \frac{1}{2}\ln\frac{1+y}{1-y}, |y| < 1, \coth^{-1} y = \frac{1}{2}\ln\frac{y+1}{y-1}, |y| > 1,$$

By the rules for differentiation we have

$$\tanh' x = \frac{1}{\cosh^2 x}$$

$$\coth' x = -\frac{1}{\sinh x}$$

By the theorem on the derivative of an inverse funnction

3 Table of Derivatives of the Basic Elementary Functions

4 Higher-order Derivative

If a function $f : E \to \mathbb{R}$ is differentiable at every point $x \in E$, then a new function $f' : E \to \mathbb{R}$ arises, whose value at a point $x \in E$ equals the derivative $f'(x)$ of the function f at that point.

The function $f' : E \to \mathbb{R}$ may itself has a derivative $(f'(x))' : E \to \mathbb{R}$ on E, called the second derivative of the original function f and denoted by one of the following two symbols:

$$f''(x), \frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}$$

and if we wish to indicate explicitly the variable of differentiation in the first case, we also write, for example, $f''_{xx}(x)$

Function $f(x)$	Derivative $f'(x)$	Restrictions on domain of $x \in \mathbb{R}$
1. C (const)	0	
2. x^{α}	$\alpha x^{\alpha-1}$	$x > 0$ for $\alpha \in \mathbb{R}$ $x \in \mathbb{R}$ for $\alpha \in \mathbb{N}$
3. a^x	$a^x \ln a$	$x \in \mathbb{R}$ $(a > 0,\, a \neq 1)$
4. $\log_a \lvert x \rvert$	$\frac{1}{x \ln a}$	$x \in \mathbb{R} \setminus 0$ $(a > 0, a \neq 1)$
5. $\sin x$	$\cos x$	
6. $\cos x$	$-\sin x$	
7. $\tan x$	$\frac{1}{\cos^2 x}$	$x \neq \frac{\pi}{2} + \pi k,\ k \in \mathbb{Z}$
8. $\cot x$	$-\frac{1}{\sin^2 x}$	$x \neq \pi k,\ k \in \mathbb{Z}$
9. $\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	$\lvert x \rvert < 1$
10. $\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$	$\lvert x \rvert < 1$
11. $\arctan x$	$\frac{1}{1+x^2}$	
12. $\operatorname{arccot} x$	$-\frac{1}{1+x^2}$	
13. $\sinh x$	$\cosh x$	
14. $\cosh x$	$\sinh x$	
15. $\tanh x$	$\frac{1}{\cosh^2 x}$	
16. $\coth x$	$-\frac{1}{\sinh^2 x}$	$x \neq 0$
17. $\operatorname{arsinh} x = \ln\left(x + \sqrt{1+x^2}\right)$	$\frac{1}{\sqrt{1+x^2}}$	
18. $\operatorname{arcosh} x = \ln\left(x \pm \sqrt{x^2-1}\right)$	$\pm\frac{1}{\sqrt{x^2-1}}$	$\lvert x \rvert > 1$
19. $\operatorname{artanh} x = \frac{1}{2} \ln \frac{1+x}{1-x}$	$\frac{1}{1-x^2}$	$\lvert x \rvert < 1$
20. $\operatorname{arcoth} x = \frac{1}{2} \ln \frac{x+1}{x-1}$	$\frac{1}{x^2-1}$	$\lvert x \rvert > 1$

Figure 2: Table of Derivatives of the Basic Elementary Functions.

Definition 4.1. By induction, if the derivative $f^{(n-1)}(x)$ of order $n-1$ of f has been defined, then the derivative of order n is defined by the formula:

$$f^{(n)}(x) = \frac{\mathrm{d}}{\mathrm{d}x} f^{(n-1)}(x)$$

The following notations are conventional for the derivative of order n:

$$\left(f^{(n)}(x)\right)(x)$$

The set of functions $f : E \to \mathbb{R}$ having continuous derivatives up to order n inclusive will be denoted as $C^{(n)}(E, \mathbb{R})$, and by the simpler symbol $C^{(n)}(E)$.

Examples 16.

$f(x)$	$f'(x)$	$f''(x)$	$\cdots$	$f^{(n)}(x)$
a^x	$a^x \ln a$	$a^x \ln^2 a$	$\cdots$	$a^x \ln^n a$
e^x	e^x	e^x	$\cdots$	e^x
$\sin x$	$\cos x$	$-\sin x$	$\cdots$	$\sin(n + n\pi/2)$
$\cos x$	$-sinx$	$-\cos x$	$\cdots$	$\cos(n + n\pi/2)$
$(1+x)^\alpha$	$\alpha(1+x)^{\alpha-1}$	$\alpha(\alpha-1)(1+x)^{\alpha-2}$	$\cdots$	$\alpha(\alpha-1)\cdots(\alpha-n+1)(1+x)^{\alpha-n}$
x^α	$\alpha x^{\alpha-1}$	$\alpha\alpha-1)x^{\alpha-2}$	$\cdots$	$\alpha(\alpha-1)\cdots(\alpha-n+1)x^{\alpha-n}$
$\log_a \lvert x \rvert$	$\frac{1}{\ln a}x^{-1}$	$\frac{-1}{\ln a}x^{-2}$	$\cdots$	$\frac{(-1)^{n-1}(n-1)!}{\ln a}x^{-n}$
$\ln \lvert x \rvert$	x^{-1}	$-x^{-2}$	$\cdots$	$(-1)^{n-1}(n-1)!x^n$

Examples 17 (Leibniz's formula). Let $u(x)$ and $v(x)$ be functions having derivatives up to order n inclusive on a common set E. The following formula of Leibniz holds for the nth derivative of their product:

$$(uv)^{(n)} = \sum_{m=0}^{n} \binom{n}{m} u^{(n-m)} v^{(m)} \tag{4}$$

Examples 18. If $P_n(x) = c_0 + c_1 x + \cdots + c_n x^n$, then

$$\begin{gathered}
P_n(0) = c_0 \\
P_n'(x) = c_1 + 2c_2 x + \cdots + nc_n x^{n-1} \Rightarrow P_n'(0) = c_1 \\
P_n''(x) = 2c_2 + 3 \cdot 2c_3 x + \cdots + n(n-1)c_n x^{n-2} \Rightarrow P_n''(0) = 2!c_2 \\
P_n^{(3)}(x) = 3 \cdot 2c_3 + \cdots + n(n-1)(n-2)c_n x^{n-3} \Rightarrow P_n^{(3)}(0) = 3!c_3 \\
\vdots \\
P_n^{(n)}(x) = n(n-1)(n-2)\cdots 2c_n \Rightarrow P_n^{(n)}(0) = n!c_n \\
P_n^{(k)} = 0 \quad \text{for} \quad k > n.
\end{gathered}$$

Thus, the polynomial $P_n(x)$ can be written as

$$P_n(x) = P_n^{(0)} + \frac{1}{1!}P_n^{(1)}(0)x + \frac{1}{2!}P_n^{(2)}(0)x^2 + \cdots + \frac{1}{n!}P_n^{(n)}(0)x^n.$$

Examples 19. Using Leibniz's formula and the fact that all the derivatives of a polynomial of order higher than the degree of the polynomial are zero, find the nth derivative of the following functions:

$$x^2 \sin x, x^2 \sinh x, x^2 \ln x, x^2 \sin x \cos x, x^2 e^x.$$

Examples 20. Let $f(x) = \arctan x$, find the values $f^{(n)}(0)(n = 1, 2, \cdots,)$

Examples 21. Let f be a differentiable function on $\mathbb{R}$. Show that

1. if f is an even function, then f' is an odd function,
2. if f is an odd function, then f' is an even function,
3. f' is odd $\Leftrightarrow$ f is even.

5 作業

5.1 解答題

設

$$f(x) = \begin{cases} x^2, & x \geq 3 \\ ax+b, & x < 3 \end{cases}$$

試確定a, b的值，使f在$x = 3$處可導。

5.2 解答題

求下列曲線在指定點處的切線，法線方程。

(1) $y = \dfrac{x^2}{4}, P(2,1)$

(2) $y = \cos x, P(0,1)$

5.3 解答題

求下列函數的導數

(1) $f(x) = |x|^3$

(2) $f(x) = \begin{cases} x+1, & x \geq 0 \\ 1, & x < 0 \end{cases}$

5.4 解答題

設函數

$$f(x) = \begin{cases} x^\alpha \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

試問：

(1) α為何值時，函數在$x = 0$點連續；

(2) α為何值時，函數在$x = 0$點可導.

5.5 求下列函數的導數

(1) $y = 3x^2 + 2$

(2) $y = \dfrac{1 - x^2}{1 + x + x^2}$

(3) $y = x^n + nx$

(4) $y = \dfrac{x}{m} + \dfrac{m}{x} + 2\sqrt{x} + \dfrac{2}{\sqrt{x}}$

(5) $y = x^3 \log_3 x$

(6) $y = e^x \cos x$

(7) $y = (x^2 + 1)(3x - 1)(1 - x^3)$

(8) $y = \dfrac{\tan x}{x}$

(9) $y = \dfrac{x}{1 - \cos x}$

(10) $y = \dfrac{1 + \ln x}{1 - \ln x}$

(11) $y = x\sqrt{1 - x^2}$

(12) $y = (x^2 - 1)^3$

(13) $y = \left(\dfrac{1 + x^2}{1 - x}\right)^3$

(14) $y = \ln(\ln x)$

(15) $y = \ln(\sin x)$

(16) $y = \ln(x + \sqrt{1 + x^2})$

(17) $y = \ln\left(\dfrac{\sqrt{1 + x} - \sqrt{1 - x}}{\sqrt{1 + x} + \sqrt{1 - x}}\right)$

(18) $y = (\sin x + \cos x)^3$

(19) $y = (\sin x^2)^3$

(20) $y = \arcsin(\sin^2 x)$

(21) $y = x^{x^x}$

(22) $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$

(23) $y = \sin(\sin(\sin x))$

(24) $y = \sin\left(\frac{x}{\sin(\frac{x}{\sin x})}\right)$

(25) $y = (x - a_1)^{a_1}(x - a_2)^{a_2} \cdots (x - a_n)^{a_n}$

5.6 求下列函數在指定點的高階導數

(1) $f(x) = 3x^3 + 4x^2 - 5x - 9$，求$f'''(1), f^{(4)}(x)(1)$

(2) $f(x) = \frac{x}{\sqrt{1+x^2}}$，求$f''(0), f''(1), f''(-1)$

5.7 求下列函數的高階導數

(1) $f(x) = x \ln x$，求$f''(x)$

(2) $f(x) = e^{-x^2}$，求$f'''(x)$

(3) $f(x) = \ln(1+x)$，求$f^{(5)}(x)$

(4) $f(x) = x^3 e^x$，求$f^{(10)}(x)$

5.8 解答題

設f為二階可導函數，求下列函數的二階導數

(1) $f(\ln x)$

(2) $f(x^n)$

(3) $f(f(x))$

5.9 解答題

求下列函數的n階導數

(1) $y = \ln x$

(2) $y = a^x (a > 0, a \neq 1)$

(3) $y = \dfrac{1}{x(1-x)}$

(4) $y = \dfrac{\ln x}{x}$

(5) $y = \dfrac{x^n}{1-x}$

5.10 解答題

求下列參數方程所確定的函數的二階導數

(1) $\begin{cases} x = a\cos^3 t \\ y = a\sin^3 t \end{cases}$

(2) $\begin{cases} x = e^t \cos t \\ y = e^t \sin t \end{cases}$