Introduction

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1 The Creation of the Calculus

> Who, by a vigor of mind almost divine, the motions and figures of the planets, the paths of comets, and the tides of seas first demonstrated. ——Newton's Epitaph

1.1 The Motivation for the Calculus

Following hard on the adoption of the function concept came the calculus, which, next to Euclidean geometry, is the greatest in all of mathematics. Though it was to some extent the answer to problems already tackled by the Greeks, the calculus was created primarily to treat the major scientific problems of the seventeenth century.

There were four major types of problems. The first was: Given the formula for the distance a body covers as a function of time, to find the velocity and acceleration at any instant; and, conversely, given the formula describing the acceleration of a body as a function of the time, to find the velocity and the distance traveled. This problem arose directly in the study of motion and the difficulty it posed was that the velocities and the acceleration of concern to the seventeenth century varied from instant to instant. In calculating an instantaneous velocity, for example, one cannot, as one can in the case of average velocity, divide the distance traveled by the time of travel, because at a given instant both the distance traveled and time are zero, and $0/0$ is meaningless. Nevertheless, it was clear on physical grounds that moving objects do have a velocity at each instant of their travel. The inverse problem of finding the distance covered, knowing the formula for velocity, involves the corresponding difficulty; one cannot multiply the velocity at any one instant by the time of travel to obtain the distance traveled because the velocity varies from instant to instant.

The second type of problem was to find the tangent to a curve. It was a problem of pure geometry, and it was of great importance for scientific applications. Optics, as we know, was one of the major scientific pursuits of the seventeenth century; the design of lenses was direct interest for Fermat, Descartes, Huygens, and Newton.

The third problem was that of finding the maximum or minimum value of a function. When a cannonball is shot from a cannon, the distance it will travel horizontally-the range-depends on the angle at which the cannon is inclined to the ground. One "practical" problem was to find the angle that would maximize the range. Early in the seventeenth century, Galileo determined that(in a vacuum) the maximum range is obtained for an angle of fire of 45 degree. The study of the motion of the planets involved maxima and minima problems, such as finding the greatest and least distances of a planet from the sun.

The fourth problem was finding the lengths of curves, for example, the distance covered by a planet in a given period of time; the area bounded by curves; volumes bounded by surfaces; centers of gravity of bodies; and the gravitational attraction that extended body, a planet for example, exerts on another body. The Greeks had used the method of exhaustion to find some areas and volumes. Despite the fact that they used it for relatively simple areas and volumes, they had to apply much ingenuity, because the method lack generality. Nor did they often come up with numerical answers. Interest in finding lengths, areas, volumes, and centers of gravity was revived when the work of Archimedes became known in Europe. The method of exhaustion was first modified gradually, and then radically by the invention of the calculus.

1.2 Early Seventeenth-Century Work on the Calculus

The problems of the calculus were tackled by at least a dozen of the greatest mathematicians of the seventeenth century and by several dozen minor ones. All of their contributions were crowned by the achievements of Newton and Leibniz. Here we shall be able to note only the principle contributions of the precursors of these two masters.

Several methods were advanced to find the tangent to a curve. Gilles Persone de Roberval(1602-75) generalized a method Archimedes had used to find the tangent at any point on his spiral, which the tangent line of a projectile shot from a canon is the resultant diagonal line of the vertical and horizontal velocities.

While the notion of a tangent as a line having the direction of the resultant velocity was more complicated than the Greek definition of a line touching a curve, this newer concept applied to many curves for which the older one failed.

2 Sets and Elementary Operations on them

In this note, we introduce some basic concepts for real analysis.

2.1 The Concept of a Set

Since the late nineteenth and early twentieth centuries the most universal language of mathematics has been the language of set theory. This is even manifest in one of the definitions of mathematics as the science that studies different structures (relations) on sets.

"We take a set to be an assemblage of definite, perfectly distinguishable objects of our intuition or our thought into a coherent whole." Thus did Georg Cantor[1], describe the concept of a set.

- A set may be consist of any distinguishable objects.
- A set is unambiguously determined by the collection of objects that comprise it.
- Any property defines the set of objects having that property.

If x is an object, P is a property, and $P(x)$ denotes the assertion that x has property P, then the class of objects having the property P is denoted $\{x|P(x)\}$

And in fact the concept of the set of all sets, for example, is simply contradictory. This is the classical paradox of **Russell**. [2]

2.2 The Inclusion Relation

The statement, "x is an element of the set X" is written briefly as

$$x \in X$$

and its negation as

$$x \notin X$$

[1]G.Cantor(1845-1918) - German mathematician, the creator of the theory of infinite sets and the progenitor of set theoretic language in mathematics.

[2]B.Russell (1872-1970) - British logician, philosopher, sociologist ans social activist.

When statements about sets are written, frequent use is made of the logical operators $\exists$ ("there exists" or "there are") and $\forall$ ("every" or "for all") which are called the *existence* and *generalization* respectively.

Thus two sets are equal if they consist of the same elements, this statement is usually written briefly as

$$A = B,$$

read as "A equals B". The negation of equality is usually written as

$$A \neq B.$$

If every element of A is an element of B, we write $A \subset B$ and say that A is a subset of B or that B contains A.

Thus

$$A \subset B := \forall x \in A \Rightarrow x \in B$$

If $A \subset B$ and $A \neq B$, we shall say that the inclusion $A \subset B$ is *strict* or that A is a proper subset of B.

Using these definitions, we can now conclude that

$$A = B \Leftrightarrow A \subset B \wedge B \subset A$$

If M is a set, any property of P distinguishes in M the subset

$$\{x \in M | P(x)\}$$

consisting of the elements of M that have the property.

For example, it is obvious that

$$M = \{x \in M | x \in M\},$$

and the *empty* subset of M is

$$\emptyset = \{x \in M | x \neq x\}$$

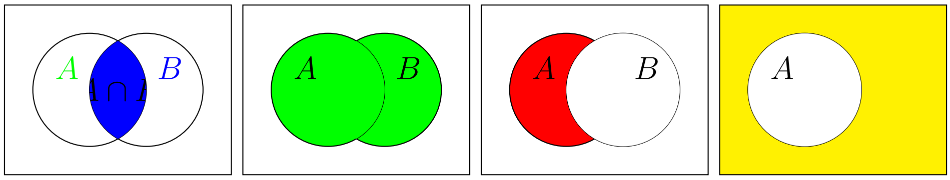
2.3 Elementary Operations on Sets

Let A and B be subsets of a set M.

(a) The **union** of A and B is the set $A \cup B \triangleq \{x \in M | x \in A \vee x \in B\}$

(b) The **intersection** of A and B is the set $A \cap B \triangleq \{x \in M \wedge x \in A \vee x \in B\}$

(c) The **difference** of A and B is the set $A \setminus B \triangleq \{x \in M | x \in A \vee x \notin B\}$



(a) (b) (c) (d)

Figure 1: (a) Intersection. (b) Union. (c) Difference. (d) Complement.

(d) The **direct(Cartesian) product of sets.** For any two sets A and B one can form a new set, namely the pair $\{A, B\} = \{B, A\}$, which consists of the sets A and B and no others.This set has two elements if $A \neq B$ and one element if $A = B$. This set is called the unordered pair of sets A and B, to be distinguished from the ordered pair (A, B) in which the elements are endowed with additional properties to distinguish the first and the second elements of the pair $\{A, B\}$. The equality

$$(A, B) = (C, D)$$

between two ordered pairs means by definition that $A = C$ and $B = D$. In particular, if $A \neq B$, then $(A, B) \neq (B, A)$.

Now let X and Y be arbitrary sets. The set

$$X \times Y \triangleq \{(x, y) | (x \in X) \wedge (y \in Y)\}$$

formed by the ordered pairs (x, y) whose first element belongs to X and whose second element belongs to Y, is called the Cartesian product of the set X and Y.

3 Functions

3.1 The Concept of a Function (Mapping)

The term *function* first appeared in the years from 1673 to 1692 in works of G.Leibniz. By the year 1698 the term had become established in a sense close to the modern one through the correspondence between Leibniz and Johann Bernoulli. [3]

[3] Johann Bernoulli (1667-1748) - one of the early representatives of the distinguished Bernoulli family of Swiss scholars, he studied analysis, geometry and mechanics. He was one of the founders of the calculus of variations. He gave the first systematic exposition of the differential and integral calculus.

Let X and Y be certain sets. We say that there is a function defined on X with values in Y if, by virtue of some rule f, to each element $x \in X$ there corresponds an element $y \in Y$. In this case the set X is called the **domain** of the function. The symbol x used to denote a general element of the domain is called the argument of the function. The element $y_0 \in Y$ corresponding to a particular value $x_0 \in X$ is called the value of the function at x_0, and is denoted as $f(x_0)$. As the argument $x \in X$ varies, the value $y = f(x) \in Y$, in general, varies depending on the values of x. For that reason, the quality $y = f(x)$ is often called the dependent variable.

The set

$$f(X) = \{y \in Y | \exists x, x \in X \wedge y = f(x)\}$$

of values assumed by a function on elements of the set X will be called the set of values or the range of the function.

For a function the following notations are standard:

$$f : X \to Y, X \xrightarrow{f} Y$$

Two functions (mapping) f_1 and f_2 are identical or equal if the have the same domain X and each element $x \in X$ the values $f_1(x)$ and $f_2(x)$ are the same. In this case we write $f_1 = f_2$.

Example 3.1. *The formulas $l = 2\pi r$ and $V = \frac{4}{3}\pi r^3$ establish functional relationships between the circumference l of a circle and its radius r and between the volume V of a ball and its radius r. Each of these formulas provides a particular function $f : \mathbb{R}_+ \to \mathbb{R}_+$ defined on the set $\mathbb{R}_+$ of the positive real numbers with values in the same set.*

Example 3.2. *The mapping $G : \mathbb{R}^2 \to \mathbb{R}^2$ (the direct product $\mathbb{R}^2 = \mathbb{R} \times mathbbR = \mathbb{R}_t \times \mathbb{R}_x$) into itself defined by the foumulas:*

$$\begin{aligned} x' &= x - vt \\ t' &= t \end{aligned}$$

is the classical Galilean transformation for transition from one inertial coordinate system (x, t) to another system (x', t') that is in motion relative to the first speed v.

The same purpose is served by the mapping $L : \mathbb{R}^2 \to \mathbb{R}^2$ defined by the relations:

$$x' = \frac{x - vt}{\sqrt{1 - \left(\frac{v}{c}\right)^2}},$$
$$t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

is the well known (one-dimensional) Lorentz transformation, which play a fundamental role in the special theory of relativity. The speed c is the speed of light.

Example 3.3. *The projection* $pr_1 : X_1 \times X_2 \to X_1$ *and* $pr_2 : X_1 \times X_2 \to X_2$ *are obvious functions.*

3.2 Elementary Classification of Mapping

A mapping $f : X \to Y$ is said to be

surjective if $f(X) = Y$;

injective if for any elements $x_1, x_2 \in X$

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

bijective if it is both surjective and injective.

3.3 Some Special Functions

Example 3.4. *The absolute value function*

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

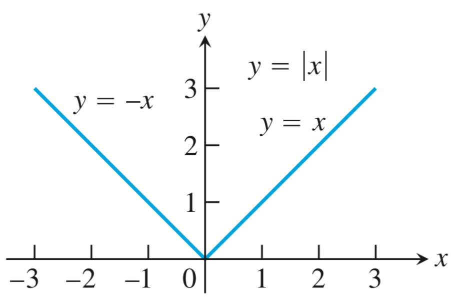


Figure 2: The absolute function.

Example 3.5. *The Greatest Integer Function This function whose value at any number x is the greatest integer less than or equal to x is called the greatest integer function or the integer floor function. It is denoted as* $\lfloor x \rfloor$

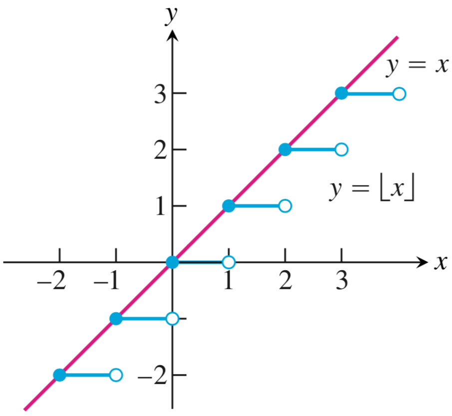


Figure 3: The greatest integer function.

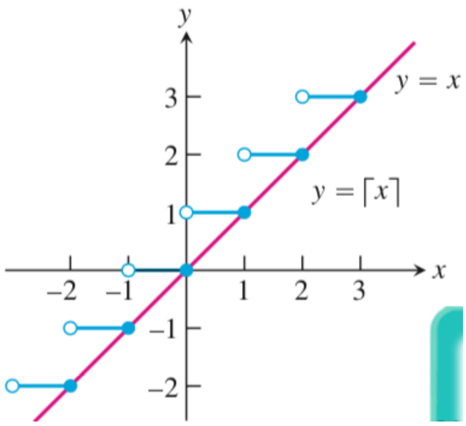


Figure 4: The least integer function.

Example 3.6. *The Least Integer Function This function whose value at any number x is the smallest integer great than or equal to x is called the leastest integer function or the integer ceiling function. It is denoted as* $\lceil x \rceil$

Example 3.7. *The Sign Function or Signum Function The signum function of a real number x is defined as follows:*

$$\mathrm{sgn(x)} = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x = 1 \end{cases}$$

Example 3.8. *The Dirichlet Function*

$$\mathrm{D(x)} = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

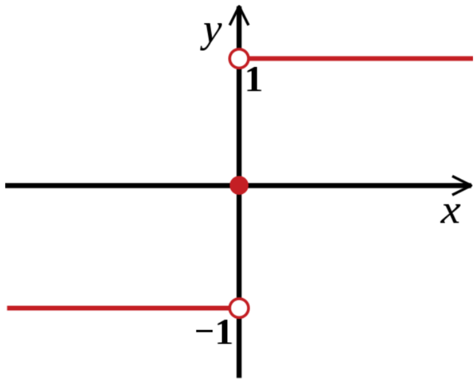


Figure 5: The singnum function.

Example 3.9. *The Riemann Function*

$$\mathrm{R(x)} = \begin{cases} \frac{1}{q}, & x = \frac{p}{q}, p, q \in \mathbb{Z}^+, (p, q) = 1 \\ 0, & x = 0, 1, (0, 1) \setminus \mathbb{Q} \end{cases}$$

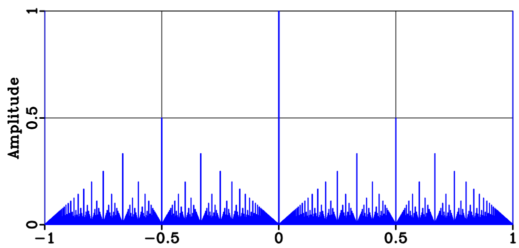


Figure 6: The Rimann function.

4 The Real Numbers

In mathematics, a real number is a value of a continuous quantity that can represent a distance along a line. The adjective **real** in this context was introduced in the 17th century by Rene **Descartes**, who distinguished between real and imaginary roots of polynomials.

The discovery of a suitable rigorous definition of the real numbers, indeed, was one of the most important developments of 19th-century mathematics. The current standard axiomatic definition is that real numbers forms the unique Dedkind-complete ordered field, up to an isomorphism, whereas popular constructive definitions of real numbers include declaring them as equivalence classes of Cauchy sequences of rational numbers, Dedekind cuts, or infinite decimal representations, together with precise interpretations for the arithmetic operations and the order relation. All these definitions satisfy the axiomatic definition and thus equivalent.

4.1 The Definition of Real Numbers

The real number system $\mathbb{R}$, can be defined axiomatically up to isomorphism, which is described hereafter. There are also many ways to construct the real number system, for example, starting from natural numbers, then defining rational numbers algebraically, and finally defining real numbers as equivalence classes of their Cauchy sequences or as Dedekind cuts, which are certain subsets of rational numbers. Another possibility is to start from some rigorous axiomatization of Euclidean geometry and then define the real number system geometrically. All these constructions are on equal footing.

4.2 The Axiom System and some General Properties of the Set of Real Numbers

Definition 4.1. *A set* $\mathbb{R}$ *is called the set of real numbers and its elements are real numbers if the following list of conditions holds, called the axiom system of the real numbers.*

(I) AXIOMS FOR ADDITION

An operation

$$+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

is defined, assigning to each ordered pair (x, y) of elements x, y of $\mathbb{R}$ a certain element $x + y \in \mathbb{R}$, called the sum of x and y. The operation satisfies the following conditions:

1. There exists a neutral, or identity element 0 (called zero) such that

   $$x + 0 = 0 + x = x$$

   for every $x \in \mathbb{R}$.

2. For every element $x \in \mathbb{R}$ there exists an element $-x \in \mathbb{R}$ called the negative of x such that

   $$x + (-x) = (-x) + x = 0$$

   .

3. The operation is associative, that is, the relation

$$x + (y + z) = (x + y) + z$$

for any elements of x, y, z of $\mathbb{R}$.

4. The operation is commutative, that is

$$x + y = y + x$$

for any elements x, y of $\mathbb{R}$.

If an operation is defined on a set G satisfying axiom 1, 2 and 3, we say that a group structure is defined on G or that G is a group. If the operation is called addition, the group is called an additive group. If it is also known that the operation is commutative, that is, condition 4 holds, the group is called commutative or *Abelian* is a group. If the operation is called addition, the group is called an additive group. If it is also known that the operation is commutative, that is, condition 4 holds, the group is called commutative or *Abelian*.

(II) AXIOMS FOR MULTIPLICATION

An operation

$$\bullet : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

(the operation of multiplication) is defined, assigning to each ordered pair (x, y) of elements of $\mathbb{R}$ a certain element $x \cdot y \in \mathbb{R}$, called the product of x and y. This operation satisfies the following conditions:

1. There exists a neutral, or identity element $1 \in \mathbb{R} \setminus 0$ (called one) such that

$$x \cdot 1 = 1 \cdot x = x$$

for every $x \in \mathbb{R}$

2. For every element $x \in \mathbb{R} \setminus 0$ there exists an element $x^{-1} \in \mathbb{R}$, called the inverse or reciprocal of x, such that

$$x \cdot x^{-1} = x^{-1} \cdot x = 1$$

3. The operation $\bullet$ is commutative, that is

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

holds for every elements x, y, z of $\mathbb{R}$.

4. The operation $\bullet$ is commutative, that is

$$x \cdot y = y \cdot x$$

for every elements x, y of $\mathbb{R}$.

(I, II) THE CONNECTION BETWEEN ADDITION AND MULTIPLICATION

Multiplication is distributive with respect to addition, that is

$$(x + y)z = xz + yz$$

for all $x, y, z \in \mathbb{R}$

We remark that if two operations satisfies these axioms are defined on a set G, then G is called a field.

(III) ORDER AXIOMS

Between elements of $\mathbb{R}$ there is a relation $\leq$, that is, for elements $x, y \in \mathbb{R}$ one can determine whether $x \leq y$ or not. Here the following conditions must hold:

1. $\forall x \in \mathbb{R}(x \leq x)$
2. $(x \leq y) \wedge (y \leq x) \Rightarrow (x = y)$
3. $(x \leq y) \wedge (y \leq z) \Rightarrow (x \leq z)$
4. $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}(x \leq y) \vee (y \leq x)$

(I, III) THE CONNECTION BETWEEN ADDITION AND ORDER ON $\mathbb{R}$

If x, y, z are elements of $\mathbb{R}$, then

$$(x \leq y) \Rightarrow (x + z) \leq (y + z)$$

(II, III) THE CONNECTION BETWEEN MULTIPLICATION AND ORDER ON $\mathbb{R}$

If x and y are elements of $\mathbb{R}$, then

$$(0 \leq x) \wedge (0 \leq y) \Rightarrow (0 \leq x \cdot y)$$

(IV) THE AXIOM OF COMPLETENESS (CONTINUITY)

If X and Y are nonempty subsets of $\mathbb{R}$ having the property that $x \leq y$ for every $x \in X$ and every $y \in Y$, then there exists $c \in \mathbb{R}$ such that $x \leq c \leq y$ for all $x \in X$ and $y \in Y$.

We now have a complete list of axioms such that any set on which those axioms hold can be considered a concrete realization or model of the real numbers.

In relation to any abstract system of axioms, at least two questions arise immediately. Fist, are these axioms consistent? That is, does there exists a set satisfying all the conditions just listed? This is the problem of consistency of the axioms. Second, does the given system of axiom determine the mathematical object uniquely? Here the uniqueness must be understood as follows. If two people A and B construct models independently, say of number systems $\mathbb{R}_A$ and $\mathbb{R}_B$, satisfying the axioms, then a bijective correspondence can be established between the system $\mathbb{R}_A$ and $\mathbb{R}_B$, say $f : \mathbb{R}_A \to \mathbb{R}_B$, preserving the arithmetic operations and the order, that is

1. $f(x + y) = f(x) + f(y)$
2. $f(x \cdot y) = f(x) \cdot f(y)$
3. $x \leq y \iff f(x) \leq f(y)$

4.3 Dedekind Cut

A Dedekind cut is a pair (A, B), where A and B are both subsets of rationals. This pair has to satisfy the following properties:

(1) A is nonempty.

(2) B is nonempty.

(3) If $a \in A$ and $c < a$ then $c \in A$.

(4) If $b \in B$ and $c > b$ then $c \in B$.

(5) If $b \notin B$ and $a < b$ then $a \in A$.

(6) If $a \notin A$ and $b > a$ then $b \in B$.

(7) For each $a \in A$ there is some $b > a$ so that $b \in A$.

(8) For each $b \in B$ there is some $a < b$ so that $a \in B$.

That's the definition. A real number is defined to be a Dedekind cut.

4.4 The Most Important Classes of Real Numbers

4.4.1 The Natural Numbers and the Principle of Mathematical Induction

Definition 4.2. *The numbers of the form* $1, 1+1, (1+1)+1,$ *and so forth are denoted respectively by* $1, 2, 3, \cdots$ *and so forth and are called natural numbers.*

Definition 4.3. *A set* $X \subset \mathbb{R}$ *is inductive if for each number* $x \in X$*, it also contains* $x + 1$.

The intersection $X = \bigcap_{\alpha \in A} X_\alpha$, if not empty, is an inductive set.

4.4.2 Rational and Irrational Numbers

a. The Integers

Definition 4.4. *The union of the set of natural numbers, the set of negatives of natural numbers, and zero is called the set of integers and is denoted* $\mathbb{Z}$.

The set $\mathbb{Z}$ is an Abelian group with respective to addition. With respective to Multiplication Z is not a group, nor is $Z \setminus 0$, since the reciprocals of the integers are not in $\mathbb{Z}$(except the reciprocal of 1 and -1).

When $k = m \cdot n^{-1} \in \mathbb{Z}$ for two integers $m, n \in \mathbb{Z}$, that is, when $m = k \cdot n$ for some $k \in \mathbb{Z}$, we say that m is divisible by n or a multiple of n, or that n is a divisor of m.

A number $p \in \mathbb{N}, p \neq 1$, is prime if it has no divisors in $\mathbb{N}$except 1 and p.

The fundamental theorem of arithmetic. Each natural number admits a representation as a product

$$n = p_1 \cdots p_k$$

where $p_1, \cdots, p_k$ are prime numbers. This representation is unique except for the order of the factors.

Numbers $m, n \in \mathbb{Z}$ are said to be relatively prime if they have no common divisor except 1 and -1.

It follows in particular from this theorem that if the product $m \cdot n$ of relatively prime numbers m and n is divisible by a prime p , then one of the two numbers is also divisible by p.

b. The Rational Numbers

Definition 4.5. *Numbers of the form $m \cdot n^{-1}$, where $m, n \in \mathbb{Z}$, are called rational.*

We denotes the set of rational numbers by $\mathbb{Q}$.

The number $q = m \cdot n^{-1}$ can also be written as a quotient [4] of m and n, that is, as a so-called rational fraction $\dfrac{m}{n}$.

c. The Irrational Numbers

Definition 4.6. *The real numbers that are not rational are called irrational.*

The classical example of an Irrational real number is $\sqrt{2}$

We shall soon see that in a certain sense nearly all real numbers are irrational. It will be shown that the cardinality of the set of the Irrational numbers is larger than that of the cardinality of the set of rational numbers and thus in fact the former equals the cardinality of the set of real numbers.

A real number is called *algebraic* if it is the root of an algebraic equation

$$a_0 x^n + \cdots a_{n-1} x + a_0 = 0$$

with rational (or equivalently, integer) coefficients. Otherwise the real number is called *transcendental*.

4.5 Basic Lemmas Connected with the Completeness of the Real Numbers

In this section we shall establish some simple useful principles, each of which could have been used as the axiom of completeness in our construction of the real numbers.

4.5.1 The nested Interval Lemma (Cauchy-Cantor Principle)

Definition 4.7. *A function $f : \mathbb{N} \to X$ of a natural-number argument is called a sequence or, more fully, a sequence of elements of X.*

The value $f(n)$ of the function f corresponding to the number $n \in \mathbb{N}$ is often denoted x_n and called the nth term of the sequence.

[4] The notation $\mathbb{Q}$ from the first letter of the English word quotient, which in turn comes from the Latin *quota*, meaning the unit part of something, and *quot*, meaning how many.

Definition 4.8. *Let $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of sets. If $X_1 \supset X_2 \supset \cdots \supset X_n \supset \cdots$, that is $X_n \supset X_{n+1}$ for all $n \in \mathbb{N}$, we say the sequence is nested.*

Lemma 4.1. *Cauchy-Cantor. For any nested sequence $I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots$ of closed intervals, there exists a point $c \in \mathbb{R}$ belonging to all these intervals.*

If in addition it is known that for any $\epsilon > 0$ there is an interval I_k whose length $|I_k|$ is less than ϵ, then c is the unique point common to all the intervals.

4.5.2 The Finite Covering Lemma (Borel-Lebesge Principle, or Heine-Borel Theorem)

Definition 4.9. *A system $S = \{X\}$ of sets X is said to cover a set Y if $Y \subset \bigcup_{X \in S} X$, (that is, if every element $y \in Y$ belongs to at least one of the sets X in the system S).*

A subset of a set $S = \{X\}$ that is a system of sets will be called a *subsystem* of S. Thus a subsystem of a system of sets is itself a system of sets of the same type.

Lemma 4.2. (Borel-Lebesge). [5] *Every system of open intervals covering a closed interval contains a finite subsystem that covers the closed interval.*

4.5.3 The Limit Point Lemma (Bolzano-Weierstras Principle)

Definition 4.10. *A point $p \in \mathbb{R}$ is a limit point of the set $X \subset \mathbb{R}$ if every neighborhood of the point contains an infinite subset of X.*

This condition is obviously equivalent to the assertion that every neighborhood of p contains at least one point of X different from p itself.

Lemma 4.3. *Every bounded infinite set of real numbers has at least one limit point.*

4.5.4 The supremum and infimum

We review the definition of the supremum and infimum and some of their properties.

[5] *É*.Borel (1871-1956) and H.Lebesgue (1875-1941) well known French mathematicians who worked in the theory of functions.

Definition 4.11. *A set $A \subset \mathbb{R}$ of real numbers is bounded from above if there exists a real number $M \in \mathbb{R}$, called an upper bound of A, such that $x \leq M$ for every $x \in A$. Similarly, A is bounded from below if there exists $m \in \mathbb{R}$, called a lower bound of A, such that $x \geq m$ for every $x \in A$. A set is bounded if it is bounded both from above and below.*

Definition 4.12. *Suppose that $A \subset \mathbb{R}$ is a set of real numbers. If $M \in \mathbb{R}$ is an upper bound of A such that $M \leq M'$ for every upper bound M' of A, then M is called the supremum of A, denoted $M = \sup A$. If $m \in \mathbb{R}$ is a lower bound of A such that $m \geq m'$ for every lower bound m' of A, then m is called the infimum of A, denoted $m = \inf A$.*

If A is not bounded from above, then we write $\sup A = \infty$, and if A is not bounded from below, we write $\inf A = -\infty$.

Proposition 4.1. *The supremum or infimum of a set A is unique if it exists. Moreover, if both exist, then $\inf A \leq \sup A$.*

Proposition 4.2. *If $A \subset \mathbb{R}$, then $M = \sup A$ if and only if (a) M is an upper bound of A; (b) for every $M' < M$, then exists $x \in A$ such that $x > M'$. Similarly, $m = \inf A$ if and only if: (a) m is a lower bound of A; (b) for every $m' > m$ there exists $x \in A$ such that $x < m'$.*

Theorem 4.4. *Every non-empty set of real numbers that is bounded from above has a supremum, and every non-empty set of real numbers that is bounded from below has an infimum.*

4.6 Countable and Uncountable Sets

4.6.1 Countable Sets

Definition 4.13. *A set X is countable if it is equivalent with the set $\mathbb{N}$ of natural numbers, that is,* $\text{card}X = \text{card}\mathbb{N}$.

Proposition 4.3. *a) An infinite subset of a countable set is countable.*

b) The union of the sets of a finite or countable system of countable sets is a countable set.

Corollary 4.5. *1)* $\text{card}\mathbb{Z} = \text{card}\mathbb{N}$

2) $\text{card}\mathbb{N}^2 = \text{card}\mathbb{N}$.

4.7 The Cardinality of the Coninuum

Definition 4.14. *The set R of real numbers is also called the number continuum, [6] and its cardinality the cardinality of the continuum.*

[6] From the Latin continuum, meaning continuous, or solid.

5 Max, Min, Sup, Inf

We would like to begin by asking for the maximum of the set $(0, 1)$. It is clear that all the elements of the set less than 1. Furthermore, 1 is the smallest number which is greater than all the values of the set $(0, 1)$.

Loosely speaking, one might say that 1 is the 'maximum value' of the set $(0, 1)$. The problem is that 1 is not a value of the set at all. In this situation, we use the word '**supremum**' instead of the word '**maximum**'. The distinction between these two concepts is described in the following.

Definition 5.1. *Let S be a set of real numbers. An upper bound for S is a number B such that $x \leq B$ for all $x \in S$. The supremum, if it exists("sup", "LUB", "least upper bound") of S is the smallest upper bound for S. An upper bound which actually belongs to the set is called a maximum.*

Proving that a certain number M is the LUB of a set S is often done in two steps:

1. Prove that M is an upper bound for S, i.e. show that $s \leq M$ for all $s \in S$.

2. Prove that M is the least upper bound for S. This is done by assuming that for any $\epsilon > 0$, such that $M - \epsilon$ is not an upper bound. In other words, there exists an element $s_0 \in S$ such that $s_0 > M - \epsilon$.

Example 5.1. *Find the least upper bound for the following set and prove that your answer is correct.*

$$S = \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots, \frac{n}{n+1}, \cdots \right\}$$

Proposition 5.1. *Suppose that M is an upper bound for a set S such that $M \in S$, then $M = \sup \{S\}$*

Example 5.2. *Find the least upper bound for the following set and prove that your answer is correct.*

$$S = \left\{ 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots, \frac{n}{n+1}, \cdots \right\}$$

Example 5.3. *Find the max, min, sup and inf of the following set and prove that your answer is correct.*

$$S = \left\{ \frac{2n+1}{n+1} | n \in N \right\}$$

The central question in this section is "Does every non-empty set of numbers have a sup?" The simple answer is No. The set $\mathcal{N}$ of natural numbers does not have a sup because it is not bound form above. "Does every set of numbers which is bounded from above have a sup?" The answer, it turns out, depends upon what we mean by the word "number". If the set is rational, the answer is No!

Now, let S be the set of all positive rational numbers r such that $r^2 < 2$, that is

$$S = \left\{ 0 < r < \sqrt{2} \,|r \in \mathcal{Q} \right\}$$

The fact that S does not have a sup in $\mathcal{Q}$ can be thought of as saying that the rational numbers do not completely fill up the number line; there is a missing number "directly to the right" of S. The fact that the set $\mathcal{R}$ of all real numbers does fill up the line is such a fundamentally important property that we take it as an axiom: the completeness axiom.

Theorem 5.1. ***Least upper Bound Axion*** *Every non-empty set of real numbers which is bounded from above has a supremum.*

Definition 5.2. *Let S be a set of real numbers. A lower bound for S is a number B such that $B \leq x$ for all $x \in S$. The infimum("inf", "GLB", "greatest lower bound") of S, if it exists, is the largest lower bound for S. A lower bound which actually belongs to the set is called a minimum.*

Theorem 5.2. ***Greatest Lower Bound Property*** *Every non-empty set of real numbers which is bounded form below has a infimum.*

Proving that a certain number M is the GLB of a set S is often done in two steps:

1. Prove that M is an lower bound for S, i.e. show that $s \geq M$ for all $s \in S$.
2. Prove that M is the largest lower bound for S. This is done by assuming that for any $\epsilon > 0$, such that $M + \epsilon$ is not an upper bound. In other words, there exists an element $s_0 \in S$ such that $s_0 < M + \epsilon$.

實數連續性：確界存在原理

實數完備性：柯西收斂原理。

確界收斂原理⇒ 單調有界原理⇒ 閉區間套定理⇒ Bolzano-Weierstrass定理⇒Cauchy收斂原理⇒ 閉區間套定理⇒ 確界原理。

連續性 $\Longleftrightarrow$ 完備性。