Mean Value Theorem

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Do not ask whether a statement is true until you know what it means.

-Errett Bishop

1 Fermat's Lemma and Rolle's Theorem

Definition 1.1. A point $x_0 \in E \subset \mathbb{R}$ is called a local maximum (resp. local minimum) and the value of a function of a function $f : E \to \mathbb{R}$ at that point a local maximum value (resp. local minimum value) if there exists a neighborhood $U_E(x_0)$ in E such that at any point $x \in U_E(x_0)$ we have $f(x) \leq f(x_0)$ (resp. $f(x) \geq f(x_0)$.

Definition 1.2. If the strict inequality $f(x) < f(x_0)$(resp. $f(x) > f(x_0)$) holds at every point $x \in U_E(x_0) \setminus x_0$, the point x_0 is called strict local maximum (resp. strict local minimum) and the value of the function $f : E \to \mathbb{R}$ a strict local maximum value (resp. strict local minimum value).

Definition 1.3. The local maxima and minima are called local extrema and the values of the function as these extreme values of the function.

Examples 1.

$$f(x) = \begin{cases} x^2 & for \quad -1 \leq x < 2 \\ 4 & for \quad 2 \leq x \end{cases}$$

Examples 2. *Let* $f(x) = \sin \frac{1}{x}$ *on set* $E = \mathbb{R} \setminus 0$.

Lemma 1.1. *(Fermat)* If a function $f : E \to \mathbb{R}$ is differentiable at an interior extremum, $x_0 \in E$, then its derivative at x_0 is $0 : f'(x_0) = 0$.

Remark. 1. Fermat's theorem thus gives a necessary condition for an interior extremum of differentiable function. For non-interior extrema it is generally not true that $f'(x_0) \neq 0$.

2. Geometrically this lemma is obvious, since it asserts that an extremum of a differentiable function the tangent to its graph is horizontal.

3. Physically this lemma means that in motion along a line the velocity must be zero at the instant when the direction reverses.

Proposition 1.1. (Rolle's Theorem[1]) If a function $f : [a, b] \to \mathbb{R}$ is continuous on a closed interval $[a, b]$ and differentiable on the open set $]a, b[$ and $f(a) = f(b)$, then there exists a point $\xi \in]a, b[$ such that $f'(\xi) = 0$.

2 The theorems of Lagrange and Cauchy on finite increments

The following proposition is one of the most frequently used and important methods of studying numerical-valued functions.

Theorem 2.1. (Lagrange's finite-increment theorem) If a function $f : [a, b] \to \mathbb{R}$ is continuous on a closed interval $[a, b]$ and differentiable on the open interval $]a, b[$, there exists a point $\xi \in]a, b[$ such that

$$f(b) - f(a) = f'(\xi)(b - a) \tag{1}$$

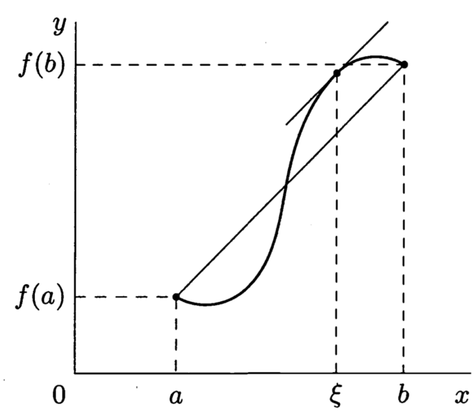

Figure 1: Lagrange's finite-increment theorem

Remark (Remark on Lagrange's Theorem). (i) In geometric language Lagrange's theorem means that at some point $(\xi, f(\xi))$, where $\xi \in]a, b[$

[1]M.Rolle(1652-1719 French mathematician)

the tangent to the graph of the function is parallel to the chord joining the points $(a, f(a))$ and $(b, f(b))$, since the slope of the chord equals $\frac{f(b)-f(a)}{b-a}$.

(ii) If x is interpreted as time and $f(b)-f(a)$ as the amount of displacement over the time $b-a$ of a particle moving along a line, Lagrange's theorem says that the velocity at some time $\xi \in]a, b[$ is the average velocity.

(iii) We note nevertheless that for motion that is not along a straight line there may be no average in the sense of Remark. Indeed, suppose the particle is moving a circle of unit radius at constant angular velocity $\omega = 1$. Its law of motion, as we know, can be written as

$$\boldsymbol{r}(t) = (\cos t, \sin t) .$$

Then

$$\dot{\boldsymbol{a}}(t) = (\cos t, \sin t)$$

and

$$|\boldsymbol{v}| = 1$$

The particle is at the same pint $\boldsymbol{r}(0) = \boldsymbol{r}(2\pi)$, the equality means that

$$\boldsymbol{r}(2\pi) - \boldsymbol{r}(0) = \boldsymbol{v}(\xi)(2\pi - 0)$$

would mean $t = 2\pi$. But this is impossible. Even so, we shall learn that there is still a relation between the displacement over a time interval and velocity. It consists of the fact that the full length of the path traversed cannot exceed the maximal absolute vale of the velocity multiplied by the time interval of the displacement.

$$|\boldsymbol{r}(b) - \boldsymbol{r}(a)| \leq \sup_{t\in[a,b]} |\dot{\boldsymbol{r}}(t)||b - a|. \tag{2}$$

As will be shown later, this natural inequality does indeed always hold. It is also called **Lagrange's finite-increment theorem**, while relation1 is often called **Lagrange mean value theorem**(the role of the mean in this case is played by both the value $f'(\xi)$ of the velocity and by the point ξ between a and b).

(iv) Lagrange's theorem is important in that it connects the increment of a function over a finite interval with the derivative. Up to now we have not had such a theorem on finite increments and have characterized only the local (infinitesimal) increment of a function in terms of the derivative or differential at a given point.

Corollary 2.2. If the derivative of a function in non-negative (resp. positive) at every point of an open interval, then the function is non-decreasing (resp. increasing) on that interval.

Corollary 2.3. A function that is continuous on a closed interval $[a, b]$ is constant on it if and only if its derivative equals zero at every point of the interval $[a, b]$(or only the open interval (a, b)).

Corollary 2.4. (Darboux theorem) Let f is differentiable on closed interval $[a, b]$, $f'_+(a) \neq f'_-(b)$ and k is a number between $f'_+(a), f'_-(b)$, then exist at least one point $\xi \in (a, b)$ such that

$$f'(\xi) = k.$$

Proposition 2.1. (Cauchy's finite-increment theorem) Let $x = x(t)$ and $y = y(t)$ be functions that are continuous on a closed interval $[a, b]$ and differentiable on the open interval $]a, b[$. Then there exists a point $\tau \in [a, b]$ such that

$$x'(\tau)(y(b) - y(a)) = y'(\tau)(x(b) - x(a))$$

If in addition $x'(t) \neq 0$ for each $t \in]a, b[$, then $x(a) \neq x(b)$ and we have the equality

$$\frac{y(b) - y(a)}{x(b) - x(a)} = \frac{y'(\tau)}{x'(\tau)}$$

Remark. (i) If we regard the pair $(x(t), y(t))$ as the law of motion of a particle, then $(x'(t), y'(t))$ is the velocity vector at time t, and $(x(\beta) - x(\alpha), y(\beta) - y(\alpha))$ is its displacement vector over the time $[\alpha, \beta]$. The theorem asserts that at some instant of time $\tau \in [\alpha, \beta]$ these two vectors are colinear.

(ii) Lagrange's formula can be obtained from Cauchy's by setting $x = x(t) = t, y(t) = y(x) = f(x), \alpha = a, \beta = b$.

3 Taylor's Formula

If we are given a function $f(x)$ having derivatives up to order n inclusive at x_0, we can immediately write the polynomial

$$P_n(x_0; x) = P_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n, \quad (3)$$

whose derivatives up to order n at the point x_0 are the same as as the corresponding derivatives of $f(x)$ at that point.

Definition 3.1. The algebraic polynomial given by Eq(3) is the Taylor polynomial of order n of $f(x)$ at x_0.

We shall be interested in the value of

$$f(x) - P_n(x_0; x) = r_n(x_0; x), \tag{4}$$

of the discrepancy between the polynomial $P_n(x)$ and the function $f(x)$, which is often called the remainder, more precisely, the nth remainder or the nth remainder term in Taylor's formula:

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + r_n(x_0; x)$$

Theorem 3.1. If the function f is continuous on the closed interval with end points x_0 and x along with its first n derivatives, and it has a derivative of order $n+1$ at the interior points of this interval, then for any function ϕ that is continuous on this closed interval and has a nonzero derivative at its interior points, there exists a point ξ between x_0 and x such that

$$r_n(x_0; x) = \frac{\phi(x) - \phi(x_0)}{\phi'(\xi)n!} f^{(n+1)}(\xi)(x - \xi)^n \tag{5}$$

Proof. On the closed interval I with endpoints x_0 and x, we consider the auxiliary function

$$F(t) = f(x) - P(t; x)$$

of the argument t. We now write out the definition of the function $F(t)$ in more detail:

$$F(t) = f(x) - \left[f(t) + \frac{f'(t)}{1!}(x - t) + \cdots + \frac{f^{(n)}(t)}{n!}(x - t)^n \right].$$

We can see that,

$$F'(t) = -\left[f'(t) - \frac{f'(t)}{1!} + \frac{f''(t)}{1!}(x - t) - \frac{f''(t)}{1!}(x - t) + \frac{f'''(t)}{2!}(x - t)^2 + \cdots + \frac{f^{(n+1)}(t)}{n!}(x - t)^n \right] = -\frac{f^{(n+1)}(t)}{n!}(x - t)^n$$

Applying Cauchy's theorem to the pair of functions $F(t), \phi(t)$ on the closed interval $[x_0, x]$,

$$\frac{F(x) - F(x_0)}{\phi(x) - \phi(x_0)} = \frac{F'(\xi)}{\phi'(\xi)}$$

□

Setting $\phi(t) = x - t$, we obtain the following corollary,

Corollary 3.2. (Cauchy's formula for the remainder term)

$$r_n(x_0; x) = \frac{1}{n!} f^{(n+1)}(\xi)(x - \xi)^n (x - x_0)$$

A particular elegant formula results if we set $\phi(t) = (x - t)^{n+1}$.

Corollary 3.3. (Lagrange's form for the remainder term)

$$r_n(x_0; x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi)(x - x_0)^{n+1}.$$

Let us consider some examples.

Examples 3. For the function $f(x) = e^x$ with $x_0 = 0$ Taylor's formula has the form

$$e^x = 1 + \frac{1}{1!}x^2 + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + r_n(0; x)$$

and the remainder is

$$r_n(0; x) = \frac{1}{(n+1)!} e^{\xi} x^{n+1}$$

where $|\xi| < |x|$. Thus

$$|r_n(0; x)| = \frac{1}{(n+1)!} e^{\xi} |x|^{n+1} < \frac{|x|^{n+1}}{(n+1)!} e^{|x|}.$$

But for each fixed $x \in \mathbb{R}$, if $n \to \infty$, the quantity $\frac{|x|^{n+1}}{(n+1)!}$, as we know , tends to zero. Hence

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + \cdots , \forall x \in \mathbb{R}$$

Examples 4. The function $a^x, 0 < a, a \neq 1$, similarly:

$$a^x = 1 + \frac{\ln a}{1!}x + \frac{\ln^2 a}{2!}x^2 + \cdots + \frac{\ln^n a}{n!}x^n + \cdots$$

Examples 5.

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots + \frac{(-1)^n}{(2n+1)!}x^{2n+1} + \cdots$$

Examples 6.

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots + \frac{(-1)^n}{(2n)!}x^{2n} + \cdots , \forall x \in \mathbb{R}$$

Examples 7.

$$\sinh x = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots + \frac{1}{(2n+1)!}x^{2n+1} + \cdots, \forall x \in \mathbb{R}$$

Examples 8.

$$\cosh x = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots + \frac{1}{(2n)!}x^{2n} + \cdots, \forall x \in \mathbb{R}$$

Examples 9. For the function $f(x) = \ln(1+x)$, we have

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots + \frac{(-1)^{(n-1)}}{n}x^n + r_n(0;x).$$

where

$$r_n(0;x) = \frac{1}{n!}\frac{(-1)^n n!}{(1+\xi)^{n+1}}(x-\xi)^n x,$$

or

$$r_n(0;x) = (-1)^n x \frac{(x-\xi)^n}{(1+\xi)^n}\frac{1}{(1+\xi)}$$

where ξ lies between 0 and x.

If $|x| < 1$, it follows from the condition that ξ lies between 0 and x that

$$\frac{|x-\xi|}{|1+\xi|} = \frac{|x|-|\xi|}{1-|\xi|} = 1 - \frac{1-|x|}{1-|\xi|} \leq 1 - \frac{1-|x|}{1-|0|} = |x|.$$

Thus for $|x| < 1$

$$|r_n(0;x)| \leq |x|^{n+1},$$

and consequently the following expansion is valid for $|x| < 1$:

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots + \frac{(-1)^n}{n}x^n + \cdots$$

Examples 10. For the function $(1+x)^\alpha$, where $\alpha \in \mathbb{R}$, we have $f^{(n)}(x) = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)(1+x)^{\alpha-n}$, so that the Talylor's formula at $x_0 = 0$ for this function has the form

$$(1+x)^\alpha = 1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!} + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + r_n(0;x) \quad (6)$$

Using Cauchy's remainder, we find

$$r_n(0;x) = \frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!}(1+\xi)^{\alpha-n-1}(x-\xi)^n x,$$

where ξ lies between x.

If $|x| < 1$,then, using the estimates, we have

$$|r_n(0;x)| \leq \left|\alpha\left(1-\frac{\alpha}{1}\right)\cdots\left(1-\frac{\alpha}{n}\right)\right|(1+\xi)^{\alpha-1}|x|^{n+1}. \qquad (7)$$

When n is increased by 1, the right side of Eq. 7 is multiplied by $\left|\left(1-\frac{\alpha}{n+1}\right)x\right|$. But since $|x| < 1$, we shall have $\left|\left(1-\frac{\alpha}{n+1}\right)x\right| < q < 1$, independently of the value α, provided $|x| < q < 1$ and n is sufficiently large.

It follows from this that $r_m(0;x) \to 0$ as $n \to \infty$ for $\alpha \in \mathbb{R}$ and any x in the open interval $|x| < 1$:

$$(1+x)^{\alpha} = 1+\frac{\alpha}{1!}x+\frac{\alpha(\alpha-1)}{2!}x^2+\cdots+\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n+\cdots \quad (8)$$

In this case $f(x) = (1+x)^n$, we write the follwoing equality:

$$(1+x)^n = 1+\frac{\alpha}{1!}x+\frac{\alpha(\alpha-1)}{2!}x^2+\cdots+\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n$$

Definition 3.2. If the function $f(x)$ has derivatives of all orders $n \in \mathbb{N}$ at a point x_0, the series

$$f(x_0)+\frac{1}{1!}f'(x_0)(x-x_0)+\cdots\frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n+\cdots$$

is called the Taylor Series of f at the point x_0.

It should not be thought that the Taylor series of an infinitely differentiable function converges in some neighborhood of x_0, for given any sequence $c_0, c_1, \cdots, c_n, \cdots$ of numbers, one can construct (although this is not simple to do) a function $f(x)$ such that $f^{(n)}(x_0) = c_n, n \in \mathbb{N}$.

It should also not be thought that if the Taylor series converges, it necessarily converges to the function that generated it. A Taylor series converges to the function that generated it only when the generating function belongs to the class of so-called **analytic function.**

Here is Cauchy's example of a non-analytic function:

$$f(x) = \begin{cases} e^{-1/x^2}, & \text{if} x \neq 0 \\ 0, & \text{if} x = 0 \end{cases} \qquad (9)$$

In conclusion, we discuss a local version of Taylor's formula. We wish to choose a polynomial $P_n(x) = x_0 + c_1(x-x_0)+\cdots+c_n(x-x_0)^n$ so as to have

$$f(x) = P_n(x) + \circ((x-x_0)^n)$$

Proposition 3.1. If there exists a polynomial $P_n(x) = c_0 + c_1(x - x_0) + \cdots + c_n(x - x_0)^n$ satisfying

$$f(x) = P_n(x) + \circ((x - x_0)^n), \tag{10}$$

that polynomial is unique.

Proof. Indeed, form Eq. 10 we see that

$$c_0 = \lim_{x \to x_0} f(x),$$

$$c_1 = \lim_{x \to x_0} \frac{f(x) - c_0}{x - x_0},$$

$$\vdots$$

$$c_n = \lim_{x \to x_0} \frac{f(x) - c_0 - c_1(x - x_0) - \cdots - c_{n-1}(x - x_0)^{n-1}}{(x - x_0)^n}$$

□

Proposition 3.2. Let E be a closed interval having $x_0 \in \mathbb{R}$ as an endpoint. If the function $f : E \to \mathbb{R}$ has derivatives $f'(x_0), f''(x_0), \cdots, f^{(n-1)}(x_0)$ up to order n inclusive at the point x_0, then the following representation holds:

$$f(x) = f'(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \circ\left((x - x_0)^n\right). \tag{11}$$

Since the Taylor polynomial $P_n(x - x_0)$ is constructed from the requirement that its derivatives up to order n inclusive must coincide with the corresponding derivatives of the function f at x_0.

Lemma 3.4. If a function $\phi : E \to \mathbb{R}$, defined on closed interval E with endpoint x_0 such that it has derivatives up to order n inclusive at x_0 and

$$\phi(x_0) = \phi'(x_0) = \cdots = \phi^{(n-1)}(x_0) = 0$$

then

$$\phi(x) = \circ\left((x - x_0)^n\right)$$

as $x \to x_0$

Let us summaries our results. We have defined the Taylor polynomial

$$P_n(x_0; x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

written the Taylor formula

$$f(x) = f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+r_n(x_0;x)$$

and obtained the following important specific form of it:

$$f(x) = f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

where ξ is a point between x_0 and x.

If f has derivatives of orders up to $n \geq 1$ inclusive at the point x_0, then

$$f(x) = f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\circ\left((x-x_0)^n\right)$$

In particular, we can now write the following table of asympototic formulas as $x \to 0$

$$e^x = 1+\frac{1}{1!}x+\frac{1}{2!}x^2+\cdots+\frac{1}{n!}x^n+\bigcirc(x^{n+1})$$

$$\cos x = 1-\frac{1}{2!}x^2+\frac{1}{4!}x^4+\cdots+\frac{(-1)^n}{(2n)!}x^{2n}+\bigcirc(x^{2n+2})$$

$$\sinh x = x+\frac{1}{3!}x^3+\frac{1}{5!}x^5+\cdots+\frac{1}{(2n+1)!}x^{2n+1}+\bigcirc(x^{2n+3})$$

$$\ln(1+x) = x-\frac{1}{2}x^2+\frac{1}{3}x^3+\cdots+\frac{(-1)^n}{n}x^n+\bigcirc(x^{n+1})$$

Examples 11. Show that $\tan x = x+\frac{1}{3}x^3+\circ(x^3)$ as $x \to 0$

Examples 12. Show that $\ln\cos x = -\frac{1}{2}x^2-\frac{1}{12}x^4-\frac{1}{45}x^6+\bigcirc(x^8)$ as $x \to 0$

Examples 13. Let us find the values of the first six derivatives of the function $\ln\cos x$ at $x=0$.

Examples 14. Let $f(x)$ be an infinitely differentiable function at the point x_0, and suppose we know the expansion

$$f'(x) = c'_0+c'_1x+\cdots+c'_nx^n+\bigcirc(x^{n+1})$$

of its derivatives in a neighborhood of zero. Then, from the uniqueness of the Taylor expansion we have

$$\left(f'(x)\right)^{(k)}(0) = k!c'_k,$$

and so $f^{(k+1)}(0) = k!c'_k$. Thus for the function $f(x)$ itself we have the expansion

$$f(x) = f(0) + \frac{c'_0}{1!}x + \frac{1!c'_1}{2!}x^2 + \cdots + \frac{n!c'_n}{(n+1)!}x^{n+1} + \bigcirc(x^{n+2}).$$

or, after simplication,

$$f(x) = f(0) + \frac{c'_0}{1}x + \frac{c'_1}{2}x^2 + \cdots + \frac{c'_n}{(n+1)}x^{n+1} + \bigcirc(x^{n+2}).$$

Examples 15. Let us find the Taylor expansion of the function $f(x) = \tan^{-1} x$ at 0.

Examples 16. Let us find the Taylor expansion of the function $f(x) = \sin^{-1} x$ at 0.

Examples 17. Find the limit $\lim\limits_{x\to 0} \dfrac{\tan^{-1} x - \sin x}{\tan x - \sin^{-1} x}$.

Examples 18. Let f be a function that is differentiable n times on an interval I. Prove the following statements.

1. If f vanishes at $n+1$ points of I, there exists a point $\xi \in I$ such that $f^{(n)}(\xi) = 0$.

2. If $x_1, x_2, \cdots x_n$ are points of the interval I, there exist a unique polynomial $L(x)$(**the Lagrange interpolation polynomial**) of degree at most $n-1$ such that $f(x_i) = L(x_i), i = 1, 2, \cdots, n$. In addition, for $x \in I$ there exist a point $\xi \in I$ such that

$$f(x) - L(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{n!} f^{(n)}(\xi).$$

3. If $x_1 < x_2 < \cdots < x_p$ are points of I and $n_i, 1 \le i \le p$, are natural numbers such that $n_1 + n_2 + \cdots + n_p = n$ and $f^{(k)}(x_i) = 0, 0 \le k \le n_i - 1$, then there exists a point ξ in the closed interval $[x_1, x_p]$ at which $f^{(n-1)}(\xi) = 0$.

4. There exists a unique polynomial $H(x)$ (the Hermite interpolating polynomial) of degree $n-1$ such that $f^{(k)}(x_i) = H^{(k)}(x_i), 0 \le k \le n_i - 1$. Moreover, inside the smallest interval containing the points x and $x_i, i = 1, 2, \cdots, p$, there is a point ξ such that

$$f(x) = H(x) + \frac{(x-x_1)^{n1}\cdots(x-x_p)^{n_p}}{n!} f^{(n)}(\xi).$$

   This formula is called the **Hermite interpolation formula**. The points $x_1, x_2, \cdots x_p$, are called the interpolation nodes of multiplicity $n_i$respectively. Special cases of the Hermite interpolation formula are the Lagrange interpolation formula.

4 The Study of Functions Using the Methods of Differential Calculus

4.1 Conditions for a Function to be Monotonic

Proposition 4.1. The following relations hold between the monotonicity properties of a function $f : E \to \mathbb{R}$ that is differentiable on an open interval $]a, b[= E$ and the sign (positivity) of its derivative f' on that interval:

$$\begin{array}{rcl} f'(x) > 0 \Rightarrow & f \text{ is increasing} & \Rightarrow f'(x) \geq 0 \\ f'(x) \geq 0 \Rightarrow & f \text{ is non-decreasing} & \Rightarrow f'(x) \geq 0 \\ f'(x) \equiv 0 \Rightarrow & f \equiv \text{ const} & \Rightarrow f'(x) \equiv 0 \\ f'(x) \leq 0 \Rightarrow & f \text{ is non-increasing} & \Rightarrow f'(x) \leq 0 \\ f'(x) < 0 \Rightarrow & f \text{ is decreasing} & \Rightarrow f'(x) \leq 0 \end{array}$$

Examples 19. Let $f(x) = x^3 - 3x + 2$ on $\mathbb{R}$. Then $f'(x) = 3x^2 - 3 = 3(x^2 - 1)$, and we can say that the function is increasing on the open interval $] - \infty, -1[$, decreasing on $] - 1, 1[$, and increasing again on $]1, +\infty[$.

4.2 Conditions for an Interior Extremum of a Function

Proposition 4.2. In order for a point x_0 to be an extremum of a funtion $f : U(x_0) \to \mathbb{R}$ defined on a neighborhood $U(x_0)$ of that point, a necessary condition is that one of the following two conditions hold: either the function is not differentiable at x_0 or $f'(x_0) = 0$.

Simple examples show that these necessary conditions are not sufficient.

Examples 20. Let $f(x) = x^3$ on $\mathbb{R}$. Then $f'(0) = 0$, but there is no extremum at x_0.

Examples 21. Let

$$f(x) = \begin{cases} x & \text{for } x > 0 \\ 2x & \text{for } x < 0 \end{cases}$$

Proposition 4.3. (Sufficient conditions for an extremum in terms of the first derivative). Let $f : U(x_0) \to \mathbb{R}$ be a function defined on a neighborhood $U(x_0)$ of the point x_0, which is continuous at the point itself and differentiable in a deleted neighborhood $U(x_0) \setminus x_0$. Let $\mathring{U}^-(x_0) = \{x \in U(x_0) | x < x_0\}$ and $\mathring{U}^+(x_0) = \{x \in U(x_0) | x > x_0\}$.

Then the following conclusions are valid:

(a) $\forall x \in \mathring{U}^-(x_0), f'(x) < 0 \wedge \forall x \in \mathring{U}^+(x_0), f'(x) < 0 \Rightarrow f$ has no extremum at x_0

(b) $\forall x \in \mathring{U}^-(x_0), f'(x) < 0 \wedge \forall x \in \mathring{U}^+(x_0), f'(x) > 0 \Rightarrow x_0$ is a strict local minimum

(c) $\forall x \in \mathring{U}^-(x_0), f'(x) > 0 \wedge \forall x \in \mathring{U}^+(x_0), f'(x) < 0 \Rightarrow x_0$ is a strict local maximum

(d) $\forall x \in \mathring{U}^-(x_0), f'(x) > 0 \wedge \forall x \in \mathring{U}^+(x_0), f'(x) > 0 \Rightarrow f$ has no extremum at x_0

Briefly, but less precisely, one can say that if the derivative changes sign in passing through the point, then the point is an extremum, while if the derivative does not change sign, the point is not an extremum.

We remark immediately, however, that these sufficient conditions are not necessary for an extremum, as one can verify using the following example:

Examples 22.

$$f(x) = \begin{cases} 2x^2 + x^2 \sin \frac{1}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

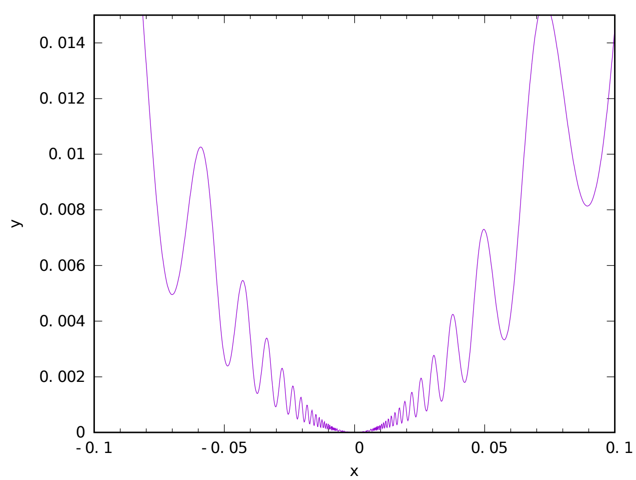

Figure 2: figure of Example 22

Since $x^2 \leq f(x) \leq 2x^2$, it is clear that the function has a strict local minimum at $x_0 = 0$.

Proposition 4.4. (Sufficient conditions for an extremum in terms of higher-order derivatives) Suppose a function $f : U(x_0) \to \mathbb{R}$ defined on a neighborhood $U(x_0)$ of x_0 has derivatives of order up to n inclusive at $x_0, (n \geq 1)$.

If $f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0$ and $f^{(n)}(x_0) \neq 0$, then there is no extrmum at x_0 if n is odd. If n is even, the point x_0 is a local extremum, in fact a strict local minimum if $f^{(n)} > 0$ and a strict local maximum if $f^{(n)} < 0$.

Examples 23. The law of refraction in geometric optics(Snell's law). According to Fermat's principle, the actual trajectory of a light ray between two points is such that the ray requires minimum time to pass from one point to the other compared with all paths joining the two points.

Examples 24. We shall show that for $x > 0$

$$x^{\alpha} - \alpha x + \alpha - 1 \leq 0, \text{ when } 0 < \alpha < 1, \tag{12}$$

$$x^{\alpha} - \alpha x + \alpha - 1 \geq 0, \text{ when } 0 > \alpha \text{ or } \alpha > 1. \tag{13}$$

Examples 25. Young's inequality If $a > 0$ and $b > 0$, and the number p and q such that $p \neq 0, 1$ and $q \neq 0, 1$, and $\dfrac{1}{p} + \dfrac{1}{q} = 1$, then

$$a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{1}{p} a + \frac{1}{q} b, \text{ if } p > 1 \tag{14}$$

$$a^{\frac{1}{p}} b^{\frac{1}{q}} \geq \frac{1}{p} a + \frac{1}{q} b, \text{ if } p < 1 \tag{15}$$

and equality holds in formula 14 and 15 only when $a = b$.

Proof. It suffices to set $x = \frac{a}{b}$ and $\alpha = \frac{1}{p}$. □

Examples 26. Holder's inequality Let $x_i, i = 1, 2, \cdots n, y_i \geq 0, i = 1, 2, \cdots, n$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Then

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} x_i^p \right)^{1/p} \left(\sum_{i=1}^{n} y_i^q \right)^{1/q} \quad \text{for } p > 1 \tag{16}$$

and

$$\sum_{i=1}^{n} x_i y_i \geq \left(\sum_{i=1}^{n} x_i^p \right)^{1/p} \left(\sum_{i=1}^{n} y_i^q \right)^{1/q} \quad \text{for } p < 1, p \neq 0 \tag{17}$$

Examples 27. Minkowski's inequality Let $x_i \geq 0, y_i \geq 0, i = 1, 2, \cdots, n$. Then

$$\left(\sum_{i=1}^{n} (x_i + y_i)^p \right)^{1/p} \leq \left(\sum_{i=1}^{n} x_i^p \right)^{1/p} + \left(\sum_{i=1}^{n} y_i^p \right)^{1/p}, \text{ when } p > 1, \tag{18}$$

$$\left(\sum_{i=1}^{n} (x_i + y_i)^p \right)^{1/p} \geq \left(\sum_{i=1}^{n} x_i^p \right)^{1/p} + \left(\sum_{i=1}^{n} y_i^p \right)^{1/p}, \text{ when } p < 1, \tag{19}$$

5 Conditions for a function to be Convex

Definition 5.1. A function $f : [a, b] \to \mathbb{R}$ defined on an open interval $]a, b[\subset \mathbb{R}$ is convex if the inequality holds

$$f(\alpha_1 x_1 + \alpha_2 x_2) \leq \alpha_1 f(x_1) + \alpha_2 f(x_2) \tag{20}$$

holds for any points $x_1, x_2 \in]a, b[$ and any numbers $\alpha_1 \geq 0, \alpha_2 \geq 0$ such that $\alpha_1 + \alpha_2 = 1$. If this inequality is strict whenever $x_1 \neq x_2$ and $\alpha_1 \alpha_2 \neq 0$ on $]a, b[$.

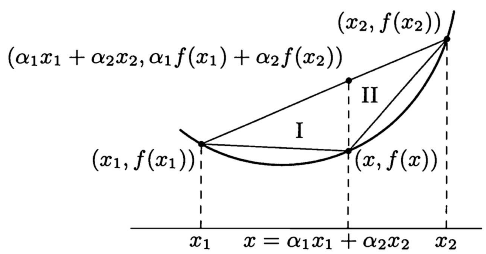

Figure 3: Convex function

Definition 5.2. If the opposite inequality holds for a function $f :]a, b[\to \mathbb{R}$, that function is said to be concave on the interval $]a, b[$, or, more often, convex upward in the interval, as opposed to a convex function, which is then said to be convex downward on $]a, b[$.

In the relations $x = \alpha_1 x_1 + \alpha_2 x_2, \alpha_1 + \alpha_2 = 1$, we have

$$\alpha_1 = \frac{x_2 - x}{x_2 - x_1}, \alpha_2 = \frac{x - x_1}{x_2 - x_1}$$

so that formula 20 can be rewritten as

$$f(x) \leq \frac{x_2 - x}{x_2 - x_1} f(x_1) + \frac{x - x_1}{x_2 - x_1} f(x_2)$$

Taking account the inequalities $x_1 < x < x_2$ and $x_1 < x_2$, we then obtain

$$(x_2 - x) f(x_1) + (x_1 - x_2) f(x) + (x - x_1) f(x_2) \geq 0$$

Remarking that $x_2 - x_1 = (x_2 - x) + (x - x_1)$ we obtain from the last inequality, after elementary transformations

$$\frac{f(x) - f(x_1)}{x - x_1} \leq \frac{f(x_2) - f(x)}{x_2 - x1} \tag{21}$$

for $x_1 < x < x_2$.

Inequality 21 is another way of writing the definition of convexity of the function $f(x)$ on an open interval $]a, b[$. Geometrically, 21 means (see Figure 3)that the slope of the chord I joining $x_1, f(x_1)$ to $x, f(x)$ is not larger than (and in the case of strict convexity is less than) the slope of the chord II joining $x, f(x)$ to $x_2, f(x_2)$.

Now let us assume that the function $f :]a, b[\to \mathbb{R}$ is differentiable on $]a, b[$. Then, letting x in Eq. 21 tend first to x_1, the tend to x_2, we obtain

$$f'(x_1) \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq f(x_2)$$

which establishes that the derivative of f is monotonic.

Taking this fact into account, for a strictly convex function we find, using Lagrange's theorem, that

$$f'(x_1) \leq f'(\xi_1) = \frac{f(x) - f(x_1)}{x - x_1} < \frac{f(x_2) - f(x)}{x_2 - x} = f'(\xi_2) \leq f'(x_2)$$

for $x_1 < \xi_1 < x < \xi_2 < x_2$, that is, strict convexity implies that the derivative is strictly monotonic.

Thus, if a differentiable function f is convex on an open interval $]a, b[$, then f' is nondecreasing on $]a, b[$, and in the case when f is strictly convex, its derivative f' is increasing on $]a, b[$.

These conditions turns out to be not only necessary, but also sufficient for convexity of a differentiable function.

Proposition 5.1. A necessary and sufficient condition for a function $f :]a, b[\to \mathbb{R}$ that is differentiable on the open interval $]a, b[$ to be convex (downward) on that interval is that its derivative f' be non-decreasing on $]a, b[$. A strictly increasing f' corresponds to a strictly convex function.

Corollary 5.1. A necessary and sufficient condition for a function $f :]a, b[\to \mathbb{R}$ that having a second derivative on the open interval $]a, b[$ to be convex on $]a, b[$ is that $f''(x) \geq 0$ on that interval. The condition $f''(x) > 0$ on $]a, b[$ is sufficient to guarantee that f is strictly convex.

Examples 28. Let us examine the convex of the following functions:

$$x^\alpha, a^x, \log_a x, \sin x$$

Proposition 5.2. A function $f :]a, b[\to \mathbb{R}$ that is differentiable on the open interval $]a, b[$ is convex(downward) on $]a, b[$ if and only if its graph contains no points below any tangent drawn to it. In that case, a necessary and sufficient condition for strict convexity is that all points of the graph except the point of tangency lie strictly above the tangent line.

Examples 29. Using the proposition to prove that

$$e^x \geq 1 + x$$

and this inequality is strict for $x \neq 0$. Similarly, using the convexity of $\ln x$, one can verify that

$$\ln x \leq x - 1$$

holds for $x > 0$, the inequality being strict for $x \neq 1$.

Definition 5.3. Let $f : U(x_0) \to \mathbb{R}$ be a function defined and differentiable on a neighborhood $U(x_0)$ of $x_0 \in \mathbb{R}$. If the function is convex downward (resp. upward) on the set $\mathring{U}^-(x_0) = \{x \in U(x_0) | x < x_0\}$ and convex upward (resp. downward) on $\mathring{U}^+(x_0) = \{x \in U(x_0) | x > x_0\}$, then $(x_0, f(x_0))$ is called an **infection point**.

An analytic criterion for the abscissa x_0 of a point of infection is easy to surmise. If $f(x)$ is twice differentiable at x_0, then since $f'(x)$ has either a maximum or minimum at x_0, we must have $f''(x_0) = 0$.

If the second derivative $f''(x)$ is defined on $U(x_0)$ and one has one sign everywhere on $\mathring{U}^-(x_0)$ and the opposite sign on $\mathring{U}^+(x_0)$,so that the point $(x_0, f(x_0))$ is a point of inflection.

Examples 30. *When consider the function $f(x) = \sin x$, we shall show that the abscissas $x = \pi k, k \in \mathbb{Z}$ are points of inflection.*

Examples 31. It should not be thought that the passing of a curve from one side of its tangent line to the other at a point is a sufficient condition for the point to be a inflection point. It may, after all, happen that the curve does not have any constant convexity on either a left or a right neighborhood of the point. A example (see Fig. 4) is

$$f(x) = \begin{cases} 2x^3 + x^3 \sin \frac{1}{x^2} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0 \end{cases}$$

Proposition 5.3. Jensen's Inequality If $f :]a, b[\to \mathbb{R}$ is a convex function, $x_1, x_2, \cdots, x_n$ are points of $]a, b[$, and $\alpha_1, \alpha_2, \cdots, \alpha_n$ are nonnegative numbers such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$, then

$$f(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n) \leq \alpha_1 f(x_1) + \alpha_2 f(x_2) + \cdots + \alpha_n f(x_n)$$

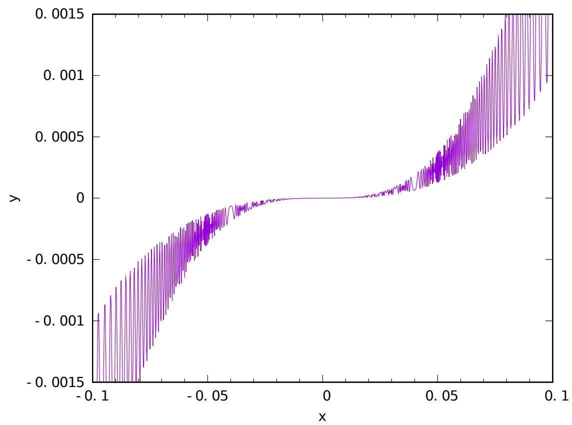

Figure 4: Figure of example 31

Examples 32. The function $f(x) = \ln x$ is strictly convex upward on the set of positive numbers. ans so we have

$$\alpha_1 \ln x_1 + \alpha_2 \ln x_2 + \cdots + \alpha_n \ln x_n \leq \ln(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n)$$

or

$$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \leq \alpha_1 x_1 + \alpha_2 x_2 + \alpha_n x_n$$

for $x_i \geq 0, i = 1, 2, \cdots, n, \sum_{i=1}^{n} \alpha_i = 1$. In partiular, if $\alpha_i = \frac{1}{n}, i = 1, 2, \cdots, n$, we obtain the classical inequality

$$\sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}$$

Examples 33. Let $f(x) = x^p, x \geq 0, p > 1$. Since such a funtion is convex, we have

$$\left(\sum_{i=1}^{n} \alpha_i x_i \right)^p \leq \sum_{i=1^n} \alpha_i x_i^p.$$

Setting $q = \dfrac{p}{p-1}, \alpha_i = b_i^q \left(\sum_{i=1}^{n} b_i^q \right)^{-1}$, and $x_i = a_i b_i^{-1/(p_i-1)} \sum_{i=1}^{n} b_i^q$ here, we obtain the Holder's inequality:

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^p \right)^{1/p} \left(\sum_{i=1}^{n} a_i^q \right)^{1/q}$$

6 L'Höspital Role

We now pause to discuss a special, but very useful device for finding the limit of a ratio of functions, known as L'Hospital [2]rule.

Proposition 6.1. Suppose the functions $f :]a, b[\to \mathbb{R}$ and $g :]a, b[\to \mathbb{R}$ are differentiable on the open interval $]a, b[$ with $g'(x) \neq 0$ on $]a, b[$ and

$$\frac{f'(x)}{g'(x)} \to A \text{ as } x \to a + 0 (-\infty \leq A \leq +\infty),$$

then

$$\frac{f(x)}{g(x)} \to A \text{ as } x \to a + 0$$

in each of the following two cases:

1. $f(x) \to 0 \wedge g(x) \to 0, \text{ as } x \to a + 0, i$
2. $g(x) \to \infty, \text{ as } x \to a + 0.$

A similar assertion holds as $x \to b - 0$.

Proof.

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(\xi)}{g'(\xi)}$$

$$\frac{f(x)}{g(x)} = \frac{f(y)}{g(x)} + \frac{f'(\xi)}{g'(\xi)} \left(1 - \frac{g(y)}{g(x)}\right)$$

□

Examples 34. Find the following limits using L'Hospital rule.

$$\lim_{x \to 0} \frac{1 - \cos 2x}{x^2}$$

$$\lim_{x \to \infty} \frac{\frac{\pi}{2} - \tan^{-1} x}{\sin \frac{1}{x}}$$

$$\lim_{x \to +\infty} \frac{x^a}{e^{bx}}, (a > 0, b > 0)$$

$$\lim_{x \to 0} x \ln x$$

$$\lim_{x \to 0^+} \ln^x \frac{1}{x}$$

[2]G.F.de l'Hospital(1661-1704), French mathematician, a capable student of Johann Bernoulli.

$$\lim_{x\to 0} x^x$$
$$\lim_{x\to\frac{\pi}{2}} \frac{1+\sin x}{1-\cos x}$$
$$\lim_{x\to\infty} \frac{x+\cos x}{x}$$

7 Asymptotic Line

Definition 7.1. The line $c_0 + c_1x$ is called the **asympotote** of the graph of the function $y = f(x)$ as $x \to -\infty$(or $x \to +\infty$) if

$$f(x) - (c_0 + c_1x) = \circ(1) \text{ as } x \to -\infty(or x \to +\infty)$$

It obviously from the Def.7.1 that

$$c_1 = \lim_{x\to-\infty} \frac{f(x)}{x}$$

and

$$c_0 = \lim_{x\to-\infty} (f(x) - c_1x)$$

In general, if $f(x) - (c_0 + c_1x + \cdots + c_nx^n) = 0$ as $x \to -\infty$, then

$$c_n = \lim_{x\to-\infty} \frac{f(x)}{x^n}$$
$$c_{n-1} = \lim_{x\to-\infty} \frac{f(x) - c_nx^n}{x^{n-1}}$$
$$\vdots$$
$$c_0 = \lim_{x\to-\infty} f(x) - (c_1x + c_2x^2 + \cdots + c_nx^n)$$

These relations, written out here for $x \to -\infty$, are of course also valid in the case $x \to +\infty$ and can be used to describe the asymptotic behavior of the graph of a function $f(x)$ using the graph of the corresponding algebraic polynomial $c_0 + c_1x + \cdots + c_nx^n$.

Examples 35. The graph of the function

$$y = x + \tan^{-1}(x^3 - 1)$$

is well approximated by the line $y = x - \frac{\pi}{2}$, as $x \to -\infty$, and by the line $y = x + \frac{\pi}{2}$, as $x \to +\infty$.

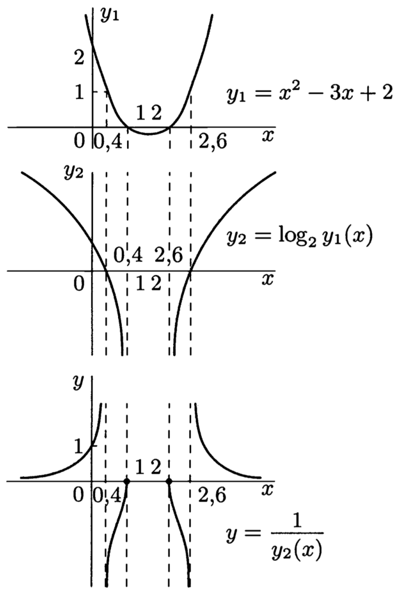

Figure 5: Graph of $\log_{x^2-3x-2} 2$

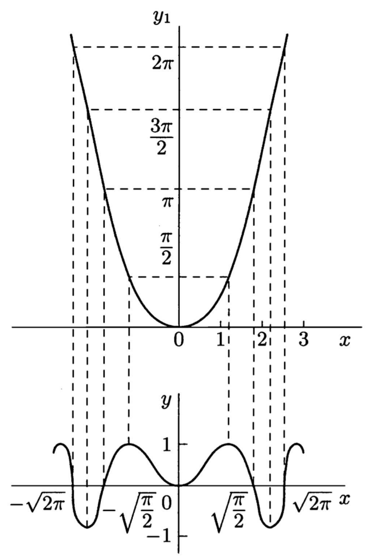

Figure 6: Graph of $\sin x^2$

7.1 Examples of Sketches of Graphs of Functions(Without Application of the Differential Calculus)

Examples 36. Let us construct a sketch of the graphs of the functions

$$h = \log_{x^2-3x-2} 2$$

$$y = \sin x^2$$

7.2 The Use of Differential Calculus in Constructing the Graph of a Function

Examples 37. Construct the graph of the function (see Figure 7)

$$f(x) = |x+2|e^{-1/x}$$

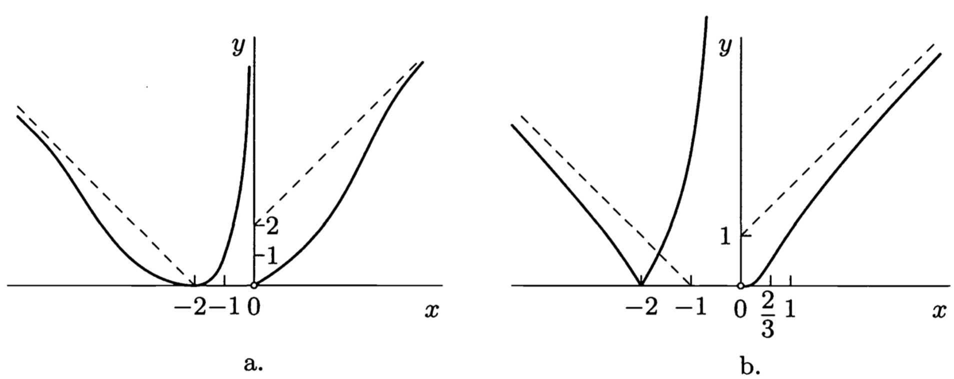

Figure 7: Graph of the function example 37

8 作業

8.1 第一部分：微分中值定理

1. 討論下列函數在指定的區間內是否存在一點，使得$f'(\xi)=0$
   (a) $f(x)=\begin{cases} x\sin\frac{1}{x}, & 0<x\leq\frac{1}{\pi} \\ 0, & x=0 \end{cases}$
   (b) $f(x)=|x|, -1\leq x\leq 1.$
2. 證明：
   (a) 方程$x^3-3x+c=0(c\in\mathbb{R})$在區間$[0,1]$內不可能有兩個不同的實根。
   (b) 方程$x^n+px+q=0(n\in\mathbb{Z}^+,p,q\in\mathbb{R})$ 當n為偶數是至多有兩個實根，當n為奇數是至多有三個實根。
3. 證明：
   (a) 若函數f在$[a,b]$上可導，且$f'(x)\geq m$，則
   $$f(b)\geq f(a)+m(b-a)$$
   (b) 若函數f在$[a,b]$上可導，且$|f'(x)|\leq M$，則
   $$|f(b)-f(a)|\leq M(b-a)$$
4. 應用Lagrange中值定理證明下列不等式：
   (a) $\dfrac{b-a}{b}<\ln\dfrac{b}{a}<\dfrac{b-a}{a}, 0<a<b.$
   (b) $\dfrac{h}{1+h^2}<\arctan h<h, h>0$
5. 應用函數的單調性證明下列不等式：
   (a) $\tan x>x-\dfrac{x^3}{3}, x\in\left(0,\dfrac{\pi}{2}\right)$
   (b) $\dfrac{2x}{\pi}<\sin x<x, x\in\left(0,\dfrac{\pi}{2}\right)$
   (c) $x-\dfrac{x^2}{2}<\ln(1+x)<x-\dfrac{x^2}{2(1+x)}, x>0$

8.2 第二部分：利用導數求極限

1. 設函數f在點a處具有二階連續導數，證明：

$$\lim_{h\to 0}\frac{f(a+h)+f(a-h)-2f(a)}{h^2}=f''(a)$$

2. 設函數f在$[a,b]$上可導，證明：存在$\xi\in(a,b)$，使得：

$$2\xi\left[f(b)-f(a)\right]=(b^2-a^2)f'(\xi)$$

3. 求下列不定極限

(1) $\lim_{x\to 0}\frac{e^x-1}{\sin x}$

(2) $\lim_{x\to\frac{\pi}{6}}\frac{1-2\sin x}{\cos 3x}$

(3) $\lim_{x\to 0}\frac{\ln(1+x)-x}{\cos x-1}$

(4) $\lim_{x\to 0}\frac{\tan x-x}{x-\sin x}$

(5) $\lim_{x\to\frac{\pi}{2}}\frac{\tan x-6}{\sec x+5}$

(6) $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$

(7) $\lim_{x\to 0}\tan x^{\sin x}$

(8) $\lim_{x\to 0}x^{\frac{1}{1-x}}$

(9) $\lim_{x\to 0}(1+x^2)^{\frac{1}{x}}$

(10) $\lim_{x\to 0_+}\sin x\ln x$

(11) $\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right)$

(12) $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$

(13) $\lim_{x\to 1}\frac{\ln\cos(x-1)}{1-\sin\frac{\pi x}{2}}$

(14) $\lim_{x\to +\infty}(\pi-2\arctan x)(\ln x)$

(15) $\lim_{x\to 0_+}x^{\sin x}$

(16) $\lim_{x\to\frac{\pi}{4}}(\tan x)^{\tan 2x}$

(17) $\lim_{x\to 0}\left(\frac{\ln(1+x)^{1+x}}{x^2}-\frac{1}{x}\right)$

(18) $\lim_{x\to 0}\left(\cot x-\frac{1}{x}\right)$

(19) $\lim_{x\to 0}\frac{(1+x)^{\frac{1}{x}}-e}{x}$

(20) $\lim_{x\to +\infty}\left(\frac{\pi}{2}-\arctan x\right)^{\frac{1}{\ln x}}$

8.3 第三部分：Talor公式

1. 求下列函數帶佩亞諾型的麥克勞林公式

   (a) $f(x)=\frac{1}{\sqrt{1+x}}$。

   (b) $f(x)=\arctan x$到含x^5的項。

   (c) $f(x)=\tan x$到含x^5的項。

2. 利用Taylor公式求下列函數的極限

   (a) $\lim_{x\to 0}\frac{e^x\sin x-x(1+x)}{x^3}$

   (b) $\lim_{x\to 0}\left[x-x^2\ln\left(1+\frac{1}{x}\right)\right]$

   (c) $\lim_{x\to 0}\frac{1}{x}\left[\frac{1}{x}-\cot x\right]$

3. 求下列函數在指定點處帶拉格朗日型余項的n階Taylor公式

   (a) $f(x)=x^3+4x^2+5$，在$x=1$處。

   (b) $f(x)=\frac{1}{1+x}$，在$x=0$處。

8.4 第四部分：單調，凹凸

1. 求下列函數的極值

   (a) $f(x)=2x^3-x^4$

   (b) $f(x)=\frac{2x}{1+x^2}$

(c) $f(x) = \dfrac{(\ln x)^2}{x}$

(d) $f(x) = \arctan x - \dfrac{1}{2}\ln(1+x^2)$

2. 證明：若函數f在x_0處滿足$f'_+(x_0) < 0(> 0), f'_-(x_0) > 0(< 0)$，則$x_0$為函數$f$的極大值(極小值)點。

3. 證明：設函數f在區間I上連續，並且在I上僅有唯一的極值點x_0，證明若x_0為f的極大（小）值點，則x_0是f在I上的最大（小）值。

4. 求下列函數在給定區間上的最大最小值

(a) $y = x^5 - 5x^4 + 5x^3 + 1, [-1, 2]$

(b) $y = 2\tan x - \tan^2 x, [0, \dfrac{\pi}{2}]$

(c) $y = \sqrt{x}\ln x, [0, +\infty]$

(d) $y = |x(x^2-1)|$

(e) $y = \dfrac{x(x^2+1)}{x^4 - x^2 + 1}$

5. 求下列函數的凹凸區間及其拐點

(a) $y = 2x^3 - 3x^2 - 36x + 25$

(b) $y = x + \dfrac{1}{x}$

(c) $y = x^2 + \dfrac{1}{x}$

(d) $y = \ln(x^2 + 1)$

(e) $y = \dfrac{1}{x^2 + 1}$

6. 應用凸函數的概念證明以下不等式

(a) 對於任意的實數$a, b, e^{\frac{a+b}{2}} \leq \dfrac{1}{2}(e^a + e^b)$

(b) 對於任何非負實數$a, b, 2\arctan\left(\dfrac{a+b}{2}\right) \geq \arctan a + \arctan b$