Primitive

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In differential calculus, as we verified on the examples of previous section, in addition to knowing how to differentiate functions and write relations between their derivatives, it is also very valuable to know how to find functions from relations satisfied by their derivatives. The simplest such problem, but, as will be seen below, a very important one, is the problem of finding a function $F(x)$ knowing its derivative $F'(x) = f(x)$.

1 The Primitive and the Indefinite Integral

Definition 1.1. A function $F(x)$ is a primitive of a function $f(x)$ on an interval if F is differentiable on the interval and satisfies the equation $F'(x) = f(x)$, or, what is the same, $\mathrm{d}F(x) = f(x)\mathrm{d}x$.

Examples 1. The function $F(x) = \tan^{-1} x$ is a primitive of the function $f(x) = \frac{1}{1+x^2}$ on the entire real line, since $(\tan^{-1} x)' = \frac{1}{1+x^2}$.

Examples 2. The function $F(x) = \cot^{-1} \frac{1}{x}$ is a primitive of $f(x) = \frac{1}{1+x^2}$ on the set of positive real numbers and on the set of negative real numbers.

Proposition 1.1. If $F_1(x)$ and $F_2(x)$ are two primitives of $f(x)$ on the same interval, then the difference $F_1(x) - F_2(x)$ is constant on that interval.

Like the operation of taking the differential, the operation of finding a primitive has the name "indefinite integration" and the mathematical notation

$$\int f(x)\mathrm{d}x \tag{1}$$

$$\mathrm{d}\int f(x)\mathrm{d}x = \mathrm{d}F(x) = f(x)\mathrm{d}x \tag{2}$$

$$\int \mathrm{d}F(x) = \int F'(x)\mathrm{d}x = F(x) + C \tag{3}$$

Formulas 2 and 3 establish a reciprocity between operations of differentiation and indefinite integration.

2 The Basic Methods of Finding a Primitive

The following rules holds:

$$\int \alpha u(x) + \beta v(x) \mathrm{d}x = \alpha \int u(x) \mathrm{d}x + \beta \int v(x) \mathrm{d}x \tag{4}$$

$$\int (uv)' \, \mathrm{d}x = \int u'v \mathrm{d}x + \int uv' \mathrm{d}x \tag{5}$$

Proposition 2.1. If $\int f(x)\mathrm{d}x = F(x) + C$, on an interval, I_x and $I_t \to I_x$ is a smooth (continuously differentiable mapping of the interval)I_x, then

$$\int (f \circ \varphi)(t)\varphi'(t)\mathrm{d}t = (F \circ \varphi)(t) + C.$$

3 Primitive of Basic Elementary Functions

$$\int x^\alpha \, \mathrm{d}x = \frac{1}{\alpha + 1} x^{\alpha+1} + C (\alpha \neq -1)$$

$$\int \frac{1}{x} \, \mathrm{d}x = \ln |x| + C$$

$$\int a^x \, \mathrm{d}x = \frac{1}{\ln a} a^x + C$$

$$\int e^x \, \mathrm{d}x = e^x + C$$

$$\int \sin x \, \mathrm{d}x = -\cos x + C$$

$$\int \cos x \, \mathrm{d}x = \sin x + C$$

$$\int \frac{1}{\cos^2 x} \, \mathrm{d}x = \tan x + C$$

$$\int \frac{1}{\sin^2 x} \, \mathrm{d}x = -\cot x + C$$

$$\int \frac{1}{\sqrt{1 - x^2}} \, \mathrm{d}x = \begin{cases} \arcsin x + C \\ -\arccos x + \widetilde{C} \end{cases}$$

$$\int \frac{1}{1 + x^2} \, \mathrm{d}x = \begin{cases} \arctan x + C \\ -\cot^{-1} x + \widetilde{C} \end{cases}$$

$$\int \sinh x \, \mathrm{d}x = \cosh x + C$$

$$\int \cosh x \, \mathrm{d}x = \sinh x + C$$

$$\int \frac{1}{\cosh^2 x} \, \mathrm{d}x = \tanh x + C$$

$$\int \frac{1}{\sinh^2 x} \, \mathrm{d}x = -\coth x + C$$

$$\int \frac{1}{\sqrt{x^2 \pm 1}} \mathrm{d}x = \ln |x + \sqrt{x^2 \pm 1}| + C$$

$$\int \frac{1}{1 - x^2} \mathrm{d}x = \frac{1}{2} \ln \left| \frac{1 + x}{1 - x} \right| + C$$

3.1 Linearity of the Indefinite Integral

Examples 3.

$$\begin{aligned}
&\int a_0 + a_1 x + \cdots + a_n x^n \mathrm{d}x \\
&= a_0 \int 1 \mathrm{d}x + a_1 \int x \mathrm{d}x + \cdots + a_n \int x^n \mathrm{d}x \\
&= a_0 x + \frac{1}{2} a_1 x^2 + \cdots + \frac{1}{n+1} a_n x^{n+1}
\end{aligned}$$

Examples 4.

$$\begin{aligned}
&\int \left(x + \frac{1}{\sqrt{x}} \right)^2 \mathrm{d}x = \int \left(x^2 + 2\sqrt{x} + \frac{1}{x} \right) \mathrm{d}x \\
&= \frac{1}{3} x^3 + \frac{4}{3} x^{3/2} + \ln |x| + C
\end{aligned}$$

3.2 Integration by Parts

$$\int u \mathrm{d}v = uv - \int v \mathrm{d}u$$

Examples 5.

$$\int \ln x \mathrm{d}x$$

Examples 6.

$$\int x^2 e^x \mathrm{d}x$$

3.3 Change of Variable in an Indefinite Integral

$$\int (f \circ \varphi)(t)\varphi'(t)\mathrm{d}t = \int f(\varphi(t))\mathrm{d}\varphi(t) = F(\varphi(t)) + C$$

Examples 7.

$$\int \frac{t}{1+t^2}\mathrm{d}t = \frac{1}{2}\int \frac{\mathrm{d}(t^2+1)}{1+t^2} = \frac{1}{2}\ln(t^2+1) + C$$

Examples 8.

$$\int \sin 2x \cos 3x \mathrm{d}x$$

Examples 9.

$$\int \sin^{-1} x \mathrm{d}x$$

Examples 10.

$$\int e^{ax} \cos bx \mathrm{d}x$$

We could have arrived at this result by using Euler's formula and the fact that the primitive of the function $e^{(a+ib)x} = e^{ax}\cos bx + ie^{ax}\sin bx$ is

$$\begin{aligned}\frac{1}{a+ib}e^{(a+ib)x} &= \frac{a-ib}{a^2+b^2}e^{(a+ib)x} \\ &= \frac{a\cos bx + b\sin bx}{a^2+b^2}e^{ax} - i\frac{a\sin bx + b\cos bx}{a^2+b^2}e^{ax}\end{aligned}$$

It should not conflate the phase "finding a primitive" with the impossible task of "expressing the primitive of a given elementary function in terms of elementary functions.

For example, the **sine integral** Si x is the primitive

$$Six = \int \frac{\sin x}{x}\mathrm{d}x$$

of the function $\frac{\sin x}{x}$ that tends to zero as $x \to 0$.

Similarly, the function

$$Cix = \int \frac{\cos x}{x}\mathrm{d}x$$

specified by the condition$Cix \to 0$ as $x \to \infty$ is not elementary. The function Cix is called the **cosine integral.**

The primitive

$$lix = \int \frac{1}{\ln x}\mathrm{d}x$$

is also not elementary. One of the primitives of this function is denoted as lix and is called **logarithmic integral.**

4 Primitive of Rational Functions

Let us consider the problem of integrating $R(x)\mathrm{d}x$, where $R(x) = \frac{P(x)}{Q(x)}$ is a ratio of polynomials.

Theorem 4.1. Every polynomial

$$P(x) = c_0 + c_1 x + \cdots + c_n x^n$$

of degree $n \geq 1$ with complex coefficients has a root in $\mathbb{C}$.

Corollary 4.2. Every polynomial

$$P(x) = c_0 + c_1 x + \cdots + c_n x^n$$

of degree $n \geq 1$ with complex coefficients admits a representation in the form

$$P(x) = c_n(x - x_1) \cdots (x - x_n)$$

where $x_1, \cdots, x_n \in \mathbb{C}$. This representation is unique up to the order of the factors.

Corollary 4.3. Every polynomial $P(x) = a_0 + a_1 x + \cdots + a_n x^n$ with real coefficients can be expanded as a product of linear and quadratic polynomials with real coefficients.

Corollary 4.4. Every root x_j of multiplicity $k_j > 1$ of polynomial $P(x)$ is a root of multiplicity $k_j - 1$ of the derivative $P'(x)$.

Corollary 4.5. a) If $Q(x) = (x - x_1)^{k_1} \cdots (x - x_p)^{k_p}$ and $\dfrac{P(x)}{Q(x)}$ is a proper fraction, there exists a unique representation of the fraction $\dfrac{P(x)}{Q(x)}$ in the form

$$\frac{P(x)}{Q(x)} = \sum_{j=1}^{p} \left(\sum_{k=1}^{k_j} \frac{a_{jk}}{(x - x_j)^k} \right)$$

b) If $P(x)$ and $Q(x)$ are polynomials with real coefficients and

$$Q(x) = (x - x_1)^{k_1} \cdots (x - x_l)^{k_l} (x^2 + p_1 x + q_1)^{m_1} \cdots (x^2 + p_n x + q_n)^{m_n}$$

there exists a unique representation of the proper fraction $\dfrac{P(x)}{Q(x)}$ in the form

$$\frac{P(x)}{Q(x)} = \sum_{j=1}^{p} \left(\sum_{k=1}^{k_j} \frac{a_{jk}}{(x - x_j)^k} \right) + \sum_{j=1}^{n} \left(\sum_{k=1}^{m_j} \frac{b_{jk} x + c_{jk}}{(x^2 + p_j x + q_j)^k} \right)$$

If we work in the domain of real numbers, then, without going outside this domain, we can express every such fraction, as we know from algebra as a sum

$$\frac{P(x)}{Q(x)} = p(x) + \sum_{j=1}^{l}\left(\sum_{k=1}^{k_j}\frac{a_{jk}}{(x-x_j)^k}\right) + \sum_{j=1}^{n}\left(\sum_{k=1}^{m_j}\frac{b_{jk}x+c_{jk}}{(x^2+p_jx+q_j)^k}\right) \tag{6}$$

We have already integrated a polynomial, so that it remains only to consider the integration of the forms

$$\int\frac{1}{(x-a)^k}\mathrm{d}x \text{ and } \int\frac{bx+c}{(x^2+px+q)^k}\mathrm{d}x$$

The first of these problems can be solved immediately, since

$$\int\frac{1}{(x-a)^k}\mathrm{d}x = \begin{cases} \frac{1}{-k+1}(x-a)^{-k+1}+C & \text{for } k\neq 1 \\ \ln|x-a|+C & \text{for } k=1 \end{cases} \tag{7}$$

With the integral

$$\int\frac{bx+c}{(x^2+px+q)^k}\mathrm{d}x$$

we proceed as follows. We present the polynomial x^2+px+q as $\left(x+\frac{1}{p}\right)^2+\left(q-\frac{1}{4}p^2\right)$, where $q-\frac{1}{4}p^2>0$, since the polynomial x^2+px+q has no real roots. Setting $x+\frac{1}{2}p=u$ and $q-\frac{1}{4}p^2=a^2$, we obtain

$$\int\frac{bx+c}{(x^2+px+q)^k}\mathrm{d}x = \int\frac{\alpha u+\beta}{(u^2+a^2)^k}\mathrm{d}u$$

where $\alpha=a, \beta=c-\frac{1}{2}bp$.

Next,

$$\int\frac{u}{(u^2+a^2)^k}\mathrm{d}u = \frac{1}{2}\int\frac{\mathrm{d}(u^2+a^2)}{(u^2+a^2)^k}$$
$$= \begin{cases} \frac{1}{2(1-k)}\left(u^2+a^2\right)^{-k+1}+C & \text{for } k\neq 1, \\ \frac{1}{2}\ln(u^2+a^2)+C & \text{for } k=1 \end{cases}$$

and it remains only to study the integral

$$I_k = \int\frac{\mathrm{d}u}{(u^2+a^2)^k}. \tag{8}$$

Integrating by parts and making elementary transformations, we have

$$\begin{aligned} I_k &= \int \frac{\mathrm{d}u}{(u^2+a^2)^k} = \frac{u}{(u^2+a^2)^2} + 2k\int \frac{u^2\mathrm{d}u}{(u^2+a^2)^{k+1}} \\ &= \frac{u}{(u^2+a^2)^k} + 2k\int \frac{(u^2+a^2)-a^2}{(u^2+a^2)^{k+1}}\mathrm{d}u = \frac{u}{(u^2+a^2)^k} + 2kI_k - 2ka^2 I_{k+1}. \end{aligned}$$

from which we obtain the recursion relation

$$I_{k+1} = \frac{1}{2ka^2}\frac{u}{(u^2+a^2)^k} + \frac{2k-1}{2ka^2}I_k \tag{9}$$

which makes it possible to lower the exponent k in the integral 8. But I_1 is easy to compute:

$$I_1 = \int \frac{\mathrm{d}u}{u^2+a^2} = \frac{1}{a}\int \frac{\mathrm{d}\left(\frac{u}{a}\right)}{1+\left(\frac{u}{a}\right)^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C. \tag{10}$$

Proposition 4.1. The primitive of any rational function $R(x) = \frac{P(x)}{Q(x)}$ can be expressed in terms of rational functions and the transcendental functions ln and $\tan^{-1}$. The rational part of the primitive, when placed over a common denominator, will have a denominator containing all the factors of the polynomial $Q(x)$ with multiplicities one less that they have in $Q(x)$.

Examples 11. Calculate $\int \frac{2x^2+5x+5}{(x^2-1)(x+2)}\mathrm{d}x$

Examples 12. Calculate $\int \frac{x^7-2x^6+4x^5-5x^4+4x^3-5x^2-x}{(x-1)^2(x^2+1)^2}\mathrm{d}x$

5 Primitive of the Form $\int R(\cos x, \sin x)\mathrm{d}x$

Let $R(u,v)$ be a rational function in u and v, that is a quotient of polynomials $\frac{P(u,v)}{Q(u,v)}$, which are linear combinations of monomials $u^m v^n$, where $m = 1, 2, \cdots, n = 1, 2, \cdots$.

Several methods exist for computing the integral $\int R(\cos x, \sin x)\mathrm{d}x$, one of which is completely general, although not always the most efficient.

a. We make the change of variable $t = \tan\frac{x}{2}$. Since

$$\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}, \quad \sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}},$$
$$\mathrm{d}t = \frac{\mathrm{d}x}{2\cos^2\frac{x}{2}}, \quad \mathrm{d}x = \frac{2\mathrm{d}t}{1+\tan^2\frac{x}{2}}.$$

as follows that

$$\int R(\cos x, \sin x)\mathrm{d}x = \int R\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)\frac{2}{1+t^2}\mathrm{d}t,$$

and the problem has been reduced to integrating a rational function.

However, this way leads to a very cumbersome rational function; for that reason one should keep in mind that in many cases there are other possibilities for rationalizing the integral.

b. In the case integral of the form $\int R(\cos^2 x, \sin^2 x)\mathrm{d}x$ or $\int r(\tan x)\mathrm{d}x$, where r is rational function, a convenient substitution is $t = \tan x$, since

$$\cos^2 x = \frac{1}{1+\tan^2 x}, \quad \sin^2 x = \frac{tan^2 x}{1+\tan^2 x}$$
$$\mathrm{d}t = \frac{\mathrm{d}x}{\cos^2 x} \Rightarrow \mathrm{d}x = \frac{\mathrm{d}t}{1+t^2}$$

Carrying out this substitution, we obtain respectively

$$\int R(\cos^2 x, \sin^2 x)\mathrm{d}x = \int R\left(\frac{1}{1+t^2}, \frac{t^2}{1+t^2}\right)\frac{\mathrm{d}t}{1+t^2}$$

$$\int r(\tan x)\mathrm{d}x = \int r(t)\frac{\mathrm{d}t}{1+t^2}$$

c. In the case of integrals of the form

$$\int R(\cos x, \sin^2 x)\sin x\mathrm{d}x, \quad \int R(\cos^2 x, \sin x)\cos x\mathrm{d}x,$$

One can move the function $\sin x$ and $\cos x$ into the differential and make the substitution $t = \cos x$ or $t = \sin x$ respectively. After these substitution, the integrals will have the form

$$-\int R(t, 1-t^2)\mathrm{d}t \text{ or } \int R(1-t^2, t)\mathrm{d}t$$

Examples 13.

$$\int \frac{\mathrm{d}x}{3+\sin x}\mathrm{d}x$$

Examples 14.

$$\int \frac{\mathrm{d}x}{(\sin x+\cos x)^2}\mathrm{d}x$$

Examples 15.

$$\int \frac{\mathrm{d}x}{2\sin^2 3x - 3\cos^2 3x + 1}\mathrm{d}x$$

Examples 16.

$$\int \frac{\cos^3 x}{\sin^7 x}\mathrm{d}x$$

6 Primitive of the Form $\int R(x, y(x))\mathrm{d}x$.

Let $R(x, y)$ be, as in previous section, a rational function. Let us consider some special integrals of the form

$$\int R(x, y(x))\mathrm{d}x$$

First of all, it is clear that if one can make a change of variable $x = x(t)$ such that both functions $x = x(t)$ and $y = y(t)$ are rational functions of t, then $x'(t)$ is also a rational function and

$$\int R(x, y(x))\mathrm{d}x = \int R(x(t), y(x(t)))x'(t)\mathrm{d}t$$

that is, the problem will have been reduced to integrating a rational function.

Consider the following special choices of the function $y = y(x)$.

a. If $y = \sqrt[n]{\dfrac{ax+b}{cx+d}}$, where $n \in \mathbb{N}$, then, setting $t^n = \dfrac{ax+b}{cx+d}$, we obtain

$$x = \frac{d \cdot t^n - b}{a - c \cdot t^n}, y = t,$$

and the integrand rationalizes.

Examples 17.

$$\int \sqrt[3]{\frac{x-1}{x+1}}\mathrm{d}x$$

b. Let us now consider the case when $y = \sqrt{ax^2+bx+c}$, that is, integrals of the form

$$\int R(x, \sqrt{ax^2+bx+c})\mathrm{d}x$$

By completing the square in the trinomial ax^2+bx+c and making a suitable liner substitution, we reduce the general case to one of the following three simple cases:

$$\int R(t, \sqrt{t^2+1})\mathrm{d}t, \int R(t, \sqrt{t^2-1})\mathrm{d}t, \int R(t, \sqrt{1-t^2})\mathrm{d}t \tag{11}$$

To rationalize these integrals it now suffices to make the following substitutions, respectively[1]:

$$\sqrt{t^2+1} = tu+1, \text{ or } \sqrt{t^2+1} = tu-1, \text{ or } \sqrt{t^2+1} = t-u;$$

$$\sqrt{t^2-1} = u(t-1), \text{ or } \sqrt{t^2-1} = u(t+1), \text{ or } \sqrt{t^2-1} = t-u;$$

$$\sqrt{1-t^2} = u(1-t), \text{ or } \sqrt{1-t^2} = u(1+t), \text{ or } \sqrt{t^2-1} = tu \pm 1.$$

Let us verify, for example, that after the first substitution we will have reduced the first integral to the integral of a rational function.

In fact, if $\sqrt{t^2+1} = tu+1$, then $t^2+1 = t^2u^2+2tu+1$, from which we find

$$t = \frac{2u}{1-u^2}$$

and then

$$\sqrt{t^2+1} = \frac{1+u^2}{1-u^2}$$

The integrals 11 can also be reduced, by means of the substitutions $t = \sinh\varphi, t = \cosh\varphi, t = \sin\varphi,$ or $t = \cos\varphi$, respectively, to the following forms:

$$\int R(\sinh\varphi, \cosh\varphi)\cosh\varphi\mathrm{d}\varphi, \int R(\cosh\varphi, \sinh\varphi)\sinh\varphi\mathrm{d}\varphi,$$
$$\int R(\sin\varphi, \cos\varphi)\cos\varphi\mathrm{d}\varphi, -\int R(\cos\varphi, \sin\varphi)\sin\varphi\mathrm{d}\varphi.$$

Examples 18.

$$\int \frac{\mathrm{d}x}{x+\sqrt{x^2+2x+1}} = \int \frac{\mathrm{d}x}{x+\sqrt{(x+1)^2+1}} = \int \frac{\mathrm{d}t}{t-1+\sqrt{t^2+1}}.$$

Proof. Setting $\sqrt{t^2+1} = u-t$. □

[1]These substitution were proposed long ago by Euler.

c.Elliptic integrals. Another important class of integrals consists of those of the form

$$\int R(x, \sqrt{P(x)})\mathrm{d}x \tag{12}$$

where $P(x)$ is a polynomial of degree of $n > 2$. As Abel and Liouville showed, such an integral cannot in general be expressed in terms of elementary function.

For $n = 3$ and $n = 4$ the integral 12 is called an **elliptic integral,** and for $n > 4$ it is called **hyperelliptic.**

It can be shown that by elementary substitutions the general elliptic integral can be reduced to the following three standard forms up to terms expressible in elementary functions:

$$\int \frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-k^2x^2)}} \tag{13}$$

$$\int \frac{x^2\mathrm{d}x}{\sqrt{(1-x^2)(1-k^2x^2)}} \tag{14}$$

$$\int \frac{\mathrm{d}x}{(1+hx^2)\sqrt{(1-x^2)(1-k^2x^2)}} \tag{15}$$

where h and k are parameters, the parameter k lying in the interval $]0,1[$ in all three cases.

By the substitution $x = \sin\varphi$ these integrals can be reduced to the following canonical integrals and combinations of them:

$$\int \frac{\mathrm{d}\varphi}{\sqrt{1-k^2\sin^2\varphi}} \tag{16}$$

$$\int \sqrt{1-k^2\sin^2\varphi}\mathrm{d}\varphi \tag{17}$$

$$\int \frac{\mathrm{d}\varphi}{(1-h\sin^2\varphi)\sqrt{1-k^2\sin^2\varphi}} \tag{18}$$

The integrals 16, 17 and 18 are called respectively the elliptic integral of **first kind**, **second kind** and **third kind**.

7 Non-elementary Special Functions

The following non-elementary special functions.

1. $Ei(x) = \int \frac{e^x}{x} \mathrm{d}x$, the exponential integral.

2. $Si(x) = \int \frac{\sin x}{x} \mathrm{d}x$, the sine integral.

3. $Ci(x) = \int \frac{\cos x}{x} \mathrm{d}x$, the cosine integral.

4. $Chi(x) = \int \frac{\cosh x}{x} \mathrm{d}x$, the hyperbolic cosine integral.

5. $Shi(x) = \int \frac{\sinh x}{x} \mathrm{d}x$, the hyperbolic sine integral.

6. $S(x) = \int \sin x^2 \mathrm{d}x$, the Fresnel integral.

7. $C(x) = \int \cos x^2 \mathrm{d}x$, the Fresnel integral.

8. $\Phi(x) = \int e^{-x^2} \mathrm{d}x$, the Euler-Poissión integral.

9. $li(x) = \int \frac{\mathrm{d}x}{\ln x}$, the logarithmic integral.

8 作業

8.1 求下列不定積分

1. $\int 1 - x + x^3 - \frac{1}{\sqrt[3]{x^2}} \, \mathrm{d}x$

2. $\int \left(x - \frac{1}{\sqrt{x}}\right)^2 \, \mathrm{d}x$

3. $\int (2^x + 3^x)^2 \, \mathrm{d}x$

4. $\int \frac{3}{\sqrt{4 - 4x^2}} \, \mathrm{d}x$

5. $\int \frac{x^2}{3(1 + x^2)} \, \mathrm{d}x$

6. $\int \tan^2 x \, \mathrm{d}x$

7. $\int \sin^2 x \, \mathrm{d}x$

8. $\int \frac{\cos 2x}{\cos x - \sin x} \, \mathrm{d}x$

9. $\int \sqrt{x\sqrt{x\sqrt{x}}} \, \mathrm{d}x$

10. $\int \left(\sqrt{\frac{1 + x}{1 - x}} + \sqrt{\frac{1 - x}{1 + x}}\right) \, \mathrm{d}x$

11. $\int (\cos x + \sin x)^2 \, \mathrm{d}x$

12. $\int \cos x \cos 2x \, \mathrm{d}x$

13. $\int \frac{x^4 + x^{-4} + 2}{x^3} \, \mathrm{d}x$

14. $\int |\sin x| \, \mathrm{d}x$

15. $\int e^{-|x|} \, \mathrm{d}x$

8.2 採用換元法或分部積分計算下列不定積分

1. $\int \cos(3x+4)\,\mathrm{d}x$

2. $\int xe^{2x^2}\,\mathrm{d}x$

3. $\int \frac{1}{2x+1}\,\mathrm{d}x$

4. $\int (1+x)^n\,\mathrm{d}x$

5. $\int \left(\frac{1}{\sqrt{3-x^2}}+\frac{1}{\sqrt{1-3x^2}}\right)\,\mathrm{d}x$

6. $\int 2^{2x+3}\,\mathrm{d}x$

7. $\int \sqrt{8-3x}\,\mathrm{d}x$

8. $\int \sqrt[3]{7-5x}\,\mathrm{d}x$

9. $\int x\sin x^2\,\mathrm{d}x$

10. $\int \frac{1}{\sin^2(2x+\pi/4)}\,\mathrm{d}x$

11. $\int \frac{1}{1+\cos x}\,\mathrm{d}x$

12. $\int \frac{1}{1+\sin x}\,\mathrm{d}x$

13. $\int \csc x\,\mathrm{d}x$

14. $\int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x$

15. $\int \frac{x}{4+x^4}\,\mathrm{d}x$

16. $\int \frac{1}{x\ln x}\,\mathrm{d}x$

17. $\int \frac{x^4}{(1-x^5)^3}\,\mathrm{d}x$

18. $\displaystyle\int \frac{x^3}{(x^8-2)}\,\mathrm{d}x$

19. $\displaystyle\int \frac{1}{x(1+x)}\,\mathrm{d}x$

20. $\displaystyle\int \cot x\,\mathrm{d}x$

21. $\displaystyle\int \cos^5 x\,\mathrm{d}x$

22. $\displaystyle\int \frac{1}{\sin x\cos x}\,\mathrm{d}x$

23. $\displaystyle\int \frac{1}{e^x+e^{-x}}\,\mathrm{d}x$

24. $\displaystyle\int \frac{2x-3}{x^2-3x+8}\,\mathrm{d}x$

25. $\displaystyle\int \frac{x^2+2}{(x+1)^3}\,\mathrm{d}x$

26. $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}}(a>0)\,\mathrm{d}x$

27. $\displaystyle\int \frac{1}{(x^2+a^2)^{3/2}}(a>0)\,\mathrm{d}x$

28. $\displaystyle\int \frac{x^5}{\sqrt{1-x^2}}\,\mathrm{d}x$

29. $\displaystyle\int \frac{\sqrt{x}}{1-\sqrt[3]{x}}\,\mathrm{d}x$

30. $\displaystyle\int \frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\,\mathrm{d}x$

31. $\displaystyle\int \arcsin x\,\mathrm{d}x$

32. $\displaystyle\int \ln x\,\mathrm{d}x$

33. $\displaystyle\int x^2\cos x\,\mathrm{d}x$

34. $\int \frac{\ln x}{x^3} \, \mathrm{d}x$

35. $\int (\ln x)^2 \, \mathrm{d}x$

36. $\int x \arctan x \, \mathrm{d}x$

37. $\int \left[\ln(\ln x) + \frac{1}{\ln x}\right] \, \mathrm{d}x$

38. $\int (\arcsin x)^2 \, \mathrm{d}x$

39. $\int (\sec x)^3 \, \mathrm{d}x$

40. $\int \sqrt{x^2 \pm a^2} \, \mathrm{d}x (a > 0)$

8.3 求下列不定積分

1. $\int [f(x)]^a f'(x) \, \mathrm{d}x (a \neq -1)$

2. $\int \frac{f'(x)}{1 + [f(x)]^2} \, \mathrm{d}x (a \neq -1)$

3. $\int \frac{f'(x)}{f(x)} \, \mathrm{d}x$

4. $\int e^{f(x)} f'(x) \, \mathrm{d}x$

8.4 求下列不定積分

1. $\int \frac{x^3}{x - 1} \, \mathrm{d}x$

2. $\int \frac{x - 1}{x^2 - 7x + 12} \, \mathrm{d}x$

3. $\int \frac{1}{1 + x^3} \, \mathrm{d}x$

4. $\int \frac{1}{1 + x^4} \, \mathrm{d}x$

5. $\displaystyle\int \frac{1}{(x-1)(x^2+1)^2}\,\mathrm{d}x$

6. $\displaystyle\int \frac{x-2}{(2x^2+2x+1)^2}\,\mathrm{d}x$

7. $\displaystyle\int \frac{1}{5-3\cos x}\,\mathrm{d}x$

8. $\displaystyle\int \frac{1}{2+\sin^2 x}\,\mathrm{d}x$

9. $\displaystyle\int \frac{1}{1+\tan x}\,\mathrm{d}x$

10. $\displaystyle\int \frac{x^2}{\sqrt{1+x-x^2}}\,\mathrm{d}x$

11. $\displaystyle\int \frac{1}{\sqrt{x^2+x}}\,\mathrm{d}x$

12. $\displaystyle\int \frac{1}{x^2}\sqrt{\frac{1-x}{1+x}}\,\mathrm{d}x$