Improper Integral

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Improper Integral and computation

Definition 0.1

Suppose that $x \to f(x)$ is defined on the interval $[a, +\infty)$ and integrable on every closed interval $[a, b]$ contained in that interval. If the limit below exists,

$$\int_a^{+\infty} f(x)\,\mathrm{d}x = \lim_{b \to +\infty} \int_a^b f(x)\,\mathrm{d}x,$$

we call it the improper Riemann integral or the improper integral of the function f over the interval $[a, +\infty)$.

Improper Integral and computation

The expression $\int_a^\infty f(x)\,\mathrm{d}x$ itself is also called an improper integral, and in that case:

1. The integral converges if the limit exists;
2. The integral diverges if the limit does not exist.

Improper Integral and computation

Definition 0.2

Suppose that $x \to f(x)$ is defined on the interval $[a, B)$ and integrable on any closed interval $[a, b] \subset [a, B)$. If the limit below exists:

$$\int_a^B f(x)\,\mathrm{d}x = \lim_{b \to B-0} \int_a^b f(x)\,\mathrm{d}x,$$

we call it the improper integral of f over the interval $[a, B)$.

Example 0.3

Let us investigate the values of the parameters α for which the integral

$$\int_0^1 \frac{1}{x^{\alpha}}\,\mathrm{d}x$$

converges.

Example 0.4

Let us investigate the values of the parameters α for which the integral

$$\int_0^{+\infty} e^{-\alpha x}\, \mathrm{d}x$$

converges.

Example 0.5

Compute the integral

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,\mathrm{d}x.$$

Example 0.6

Let us investigate the integral

$$\int_{-\infty}^{+\infty} \frac{e^{\frac{1}{x}}}{x^2}\,\mathrm{d}x.$$

Example 0.7

Compute the integral

$$\int_0^1 \ln x \, \mathrm{d}x.$$

Example 0.8

Compute the integral

$$\int_0^{+\infty} e^{-x} x^n \,\mathrm{d}x.(n \in \mathbb{Z}^+)$$

Example 0.9

Compute the integral

$$\int_0^{\frac{\pi}{2}} \ln \sin x \, \mathrm{d}x.$$

Example 0.10

Compute the integral

$$\int_0^{+\infty} \frac{1}{(1+x^2)(1+x^\alpha)}\,\mathrm{d}x.$$

Convergence of an Improper Integral

Let $[a, \omega)$ be a finite or infinite interval and $x \to f(x)$ a function defined on that interval and integrable over every closed interval $[a, b] \subset [a, \omega)$. Then by definition

$$\int_a^\omega f(x)\,\mathrm{d}x = \lim_{b \to \omega} \int_a^b f(x)\,\mathrm{d}x, \tag{1}$$

if this limit exists as $b \to \omega, b \in [a, \omega)$.

The convergence of the improper integral $\int_a^\omega f(x)\,\mathrm{d}x$ is equivalent to the existence of a limit for the function

$$\mathcal{F}(b) = \int_a^b f(x)\,\mathrm{d}x \tag{2}$$

as $b \to \omega$, $b \in [a, \omega)$.

Convergence of an Improper Integral

Theorem 0.11

If the function $x \to f(x)$ is defined on the interval $[a, \omega)$ and integrable on every closed interval $[a, b] \subset [a, \omega)$, then the integral $\int_a^\omega f(x)\,\mathrm{d}x$ converges if and only is for every $\epsilon > 0$ there exists $B \in [a, \omega)$, such that the relation

$$\left| \int_{b_1}^{b_2} f(x)\,\mathrm{d}x \right| \leq \epsilon$$

for any $b_1, b_2 \in [a, \omega)$ satisfying $B < b_1$ and $B < b_2$.

Absolute Convergence of an Improper Integral

Definition 0.12

The improper integral $\int_a^\omega f(x)\,\mathrm{d}x$ converges absolutely if the integral $\int_a^\omega |f|\,\mathrm{d}x$ converges.

Absolute Convergence of an Improper Integral

Theorem 0.13

If a function $f \geq 0$ and integrable on every $[a, b] \subset [a, \omega)$, then the improper integral $\int_a^{\omega} f(x)\,\mathrm{d}x$ exists if and only if the function $\mathcal{F}(b) = \int_a^b f(x)\,\mathrm{d}x$ is bounded on $[a, \omega)$.

Absolute Convergence of an Improper Integral

Theorem 0.14

Suppose that the function $x \to f(x)$ and $x \to g(x)$ are defined on the interval $[a, \omega)$ and integrable on any closed interval $[a, b] \subset [a, \omega)$. If

$$0 \leq f(x) \leq g(x)$$

on $[a, \omega)$, then the convergence of $\int_a^\omega g(x)\,\mathrm{d}x$ implies convergence of $\int_a^\omega f(x)\,\mathrm{d}x$, and the inequality

$$\int_a^\omega f(x)\,\mathrm{d}x \leq \int_a^\omega g(x)\,\mathrm{d}x$$

holds. Divergence of the integral $\int_a^\omega f(x)\,\mathrm{d}x$ implies divergence of $\int_a^\omega g(x)\,\mathrm{d}x$.

Absolute Convergence of an Improper Integral

Example 0.15

Let us discuss the integral

$$\int_0^{+\infty} \frac{\sqrt{x}}{\sqrt{1+x^4}}\,\mathrm{d}x$$

Absolute Convergence of an Improper Integral

Example 0.16

Let us discuss the integral

$$\int_1^{+\infty} \frac{\cos x}{x^2}\,\mathrm{d}x$$

Absolute Convergence of an Improper Integral

Example 0.17

Let us discuss the integral

$$\int_1^{+\infty} e^{-x^2}\,\mathrm{d}x$$

Absolute Convergence of an Improper Integral

Example 0.18

Let us discuss the integral

$$\int_e^{+\infty} \frac{1}{\ln x} \, \mathrm{d}x$$

Absolute Convergence of an Improper Integral

Example 0.19

Let us discuss the Euler integral

$$\int_0^{\frac{\pi}{2}} \ln \sin x \, \mathrm{d}x$$

Absolute Convergence of an Improper Integral

Example 0.20

Let us discuss the elliptic integral

$$\int_0^1 \frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}}\,\mathrm{d}x (0 < k^2 < 1)$$

Absolute Convergence of an Improper Integral

Example 0.21

Let us discuss the integral

$$\int_0^1 \frac{1}{\cos\theta - \cos\varphi}\,\mathrm{d}x$$

Conditional Convergence of an Improper Integral

Definition 0.22

If an improper integral converges but not absolutely, we say that it converges conditionally.

Conditional Convergence of an Improper Integral

Example 0.23

The integral

$$\int_{\frac{\pi}{2}}^{+\infty} \frac{\sin x}{x} \,\mathrm{d}x = -\left.\frac{\cos x}{x}\right|_{\frac{\pi}{2}}^{+\infty} - \int_{\frac{\pi}{2}}^{+\infty} \frac{\cos x}{x^2} \,\mathrm{d}x = -\int_{\frac{\pi}{2}}^{+\infty} \frac{\cos x}{x^2} \,\mathrm{d}x$$

Conditional Convergence of an Improper Integral

Theorem 0.24

Let $x \to f(x)$ and $x \to g(x)$ be functions defined on an interval $[a, \omega)$ and integrable on every closed interval $[a, b] \subset [a, \omega)$. Suppose that g is monotonic. Then a sufficient condition for convergence of the improper integral

$$\int_a^{\omega} (fg)\, \mathrm{d}x$$

is that the one of the following pairs of conditions hold:

1.
   1. the integral $\int_a^{\omega} f(x)\, \mathrm{d}x$ converges;
   2. the function g is bound on $[a, \omega)$.
2.
   1. the function $\mathcal{F}(b) = \int_a^{b} f(x)\, \mathrm{d}x$ is bound on $[a, \omega)$;
   2. the integral $g(x)$ converges to zero as $x \to \omega, x \in [a, \omega)$.

Conditional Convergence of an Improper Integral

Example 0.25

Let us discuss the **Euler-Possion** integral

$$\int_{-\infty}^{+\infty} e^{-x^2}\,\mathrm{d}x$$

Conditional Convergence of an Improper Integral

Example 0.26

Let us discuss the integral

$$\int_0^{+\infty} \frac{1}{x^\alpha} \, \mathrm{d}x$$

Conditional Convergence of an Improper Integral

Example 0.27

Let us discuss the integral

$$\int_0^{+\infty} \frac{\sin x}{x^\alpha}\,\mathrm{d}x$$

The last slide!