Integration

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Content

Definition of Integral

Definition 0.1

If a function f is defined on the closed interval $[a, b]$ and (P, ξ) is a partition with distinguished points on this closed interval, the sum

$$\sigma(f; P, \xi) = \sum_{i=1}^{n} f(\xi_i)\Delta x_i, \tag{1}$$

where $\Delta x_i = x_i - x_{i-1}$, is the **Riemann sum** of the function f corresponding to the partition (P, ξ) with distinguished point on $[a, b]$.

Definition of Integral

Definition 0.2

A function f is Riemann integrable on the closed interval $[a, b]$ if the limit of the Riemann sum

$$\sigma(f; P, \xi) = \sum_{i=1}^{n} f(\xi_i)\Delta x_i, \tag{2}$$

exists, as $\lambda(P) \to 0$(that is, the Riemann integral of f is defined).

Definition of Integral

Theorem 0.3

A necessary condition for a function f defined on a closed interval $[a, b]$ to be Riemann integrable on $[a, b]$ is that f be bounded on $[a, b]$.

Definition of Integral

Theorem 0.4

A sufficient condition for a bounded function f to be integrable on a closed interval $[a, b]$ is that for every $\epsilon > 0$ there exist a number $\delta > 0$ such that

$$\sum_{i=0}^{n} \omega(f; \Delta_i)\Delta x_i < \epsilon$$

for any partition P of $[a, b]$ with mesh $\lambda(P) < \delta$.

Definition of Integral

Corollary 0.5

$(f \in C[a, b]) \Rightarrow (f \in \mathcal{R}[a, b])$, *that is, every continuous function on a closed interval is integrable on that close interval.*

Corollary 0.6

If a bounded function f *on a closed interval* $[a, b]$ *is continuous everywhere except at a finite set of points, then* $f \in \mathcal{R}[a, b]$.

Corollary 0.7

A monotonic function on a closed interval is integrable on that interval.

Definition of Integral

Theorem 0.8

A bounded real-valued function $f : [a, b] \to \mathbb{R}$ is Rimann integrable on $[a, b]$ if and only if the following limit exist and are equal to each other:

$$\underline{I} = \lim_{\lambda(P)\to 0} s(f; P); \overline{I} = \lim_{\lambda(P)\to 0} S(f; P). \tag{3}$$

When this happens, the common value $I = \underline{I} = \overline{I}$ is the integral

$$\int_a^b f(x)\, \mathrm{d}x$$

Definition of Integral

Theorem 0.9

A necessary and sufficient condition for a function $f : [a, b] \to \mathbb{R}$ defined on a closed interval $[a, b]$ to be **Riemann integrable** on $[a, b]$ is the following relation:

$$\lim_{\lambda(P)\to 0} \sum_{i=1}^{n} \omega(f; \Delta_i)\Delta x_i = 0 \tag{4}$$

The Vector Space $\mathcal{R}[a, b]$

Theorem 0.10

If $f, g \in \mathcal{R}[a, b]$, then

1. $(f + g) \in \mathcal{R}[a, b]$;
2. $\alpha f \in \mathcal{R}[a, b]$, *where α is a numerical coefficient;*
3. $|f| \in \mathcal{R}[a, b]$;
4. $f|_{[c,d]} \in \mathcal{R}[a, b]$ *if* $[c, d] \subset [a, b]$;
5. $(f \cdot g) \in \mathcal{R}[a, b]$.

Linearity, Additivity and Monotonicity of the Integral

Theorem 0.11

If $f, g \in \mathcal{R}[a,b]$, *then* $\alpha f + \beta g \in \mathcal{R}[a,b]$, *and*

$$\int_a^b (\alpha f + \beta g)\ \mathrm{d}x = \alpha \int_a^b f(x)\,\mathrm{d}x + \beta \int_a^b g(x)\,\mathrm{d}x$$

Theorem 0.12

if $a < b < c$ *and* $f \in \mathcal{R}[a,c]$, *then* $f|_{[a,b]} \in \mathcal{R}[a,b], f|_{[b,c]} \in \mathcal{R}[b,c]$, *and the following equality holds:*

$$\int_a^c f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x$$

Linearity, Additivity and Monotonicity of the Integral

Theorem 0.13

Let $a, b, c \in \mathbb{R}$ and let f be a function integrable over the largest closed interval having two of these points as endpoints. Then the restriction of f to each of the other closed interval is also integrable over those intervals and the following equality holds:

$$\int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x + \int_c^a f(x)\,\mathrm{d}x = 0$$

A General Estimation of the Integral.

Theorem 0.14

If $a \leq b$ *and* $f \in \mathcal{R}[a,b]$, *then* $|f| \in \mathcal{R}[a,b]$ *and the following inequality holds*

$$\left| \int_a^b f(x)\,\mathrm{d}x \right| \leq \int_a^b |f|(x)\,\mathrm{d}x$$

If $|f|(x) \leq C$ *on* $[a,b]$ *then*

$$\int_a^b |f|\,\mathrm{d}x \leq C(b-a)$$

Theorem 0.15

If $a \leq b, f_1, f_2 \in \mathcal{R}[a,b], f_1(x) \leq f_2(x), \forall x \in [a,b]$, *then*

$$\int_a^b f_1(x)\,\mathrm{d}x \leq \int_a^b f_2(x)\,\mathrm{d}x$$

A General Estimation of the Integral.

Corollary 0.16

If $a \le b, f \in \mathcal{R}[a,b], m \le f(x) \le M, \forall x \in [a,b]$, *then*

$$m(b-a) \le \int_a^b f(x)\,\mathrm{d}x \le M(b-a)$$

Corollary 0.17

If $a \le b, f \in \mathcal{R}[a,b], m = \int_{x\in[a,b]} f(x), M = \sup_{x\in[a,b]} f(x)$, *then there exists a number* $\mu \in [m,M]$ *such that*

$$\int_a^b f(x)\,\mathrm{d}x = \mu(b-a)$$

A General Estimation of the Integral.

Corollary 0.18

If $f \in C[a, b]$*, there exists a point* $\xi \in [a, b]$ *such that*

$$\int_a^b f(x)\,\mathrm{d}x = f(\xi)(b - a) \tag{5}$$

A General Estimation of the Integral.

Theorem 0.19 (First Mean-Value Theorem)

Let $f, g \in \mathcal{R}[a,b]$, $m = \inf_{x\in[a,b]} f(x)$, $M = \sup_{x\in[a,b]} f(x)$. *If* g *is nonnegative (or nonpositive) on* $[a,b]$, *then*

$$\int_a^b (fg)\,\mathrm{d}x = \mu \int_a^b g(x)\,\mathrm{d}x$$

where $\mu \in [m, M]$ *If, in addition, it is known that* $f \in C[a,b]$, *then there exits a point* $\xi \in [a,b]$ *such that*

$$\int_a^b (fg)\,\mathrm{d}x = f(\xi) \int_a^b g(x)\,\mathrm{d}x$$

A General Estimation of the Integral.

Theorem 0.20 (Second mean-value theorem for the integral)

If $f, g \in \mathcal{R}[a, b]$ and g is a monotonic function on $[a, b]$, then there exists a point $\xi \in [a, b]$ such that

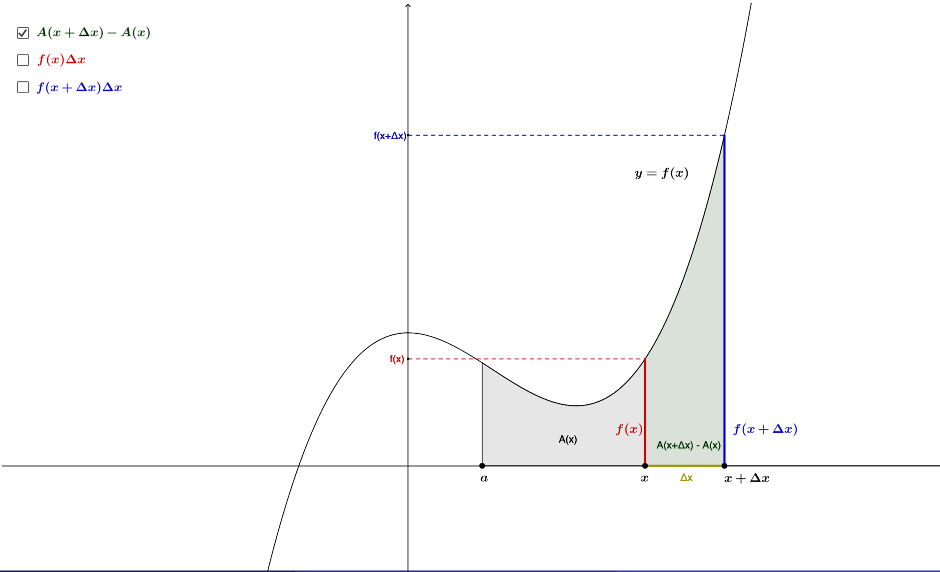
$$\int_a^b (fg)\,\mathrm{dx} = g(a)\int_a^{\xi} f(x)\,\mathrm{dx} + g(b)\int_{\xi}^{b} f(x)\,\mathrm{dx}$$

Let f be a Riemann-integrable function on a closed interval $[a, b]$. On this interval let us consider the function

$$F(x) = \int_a^x f(t)\,\mathrm{d}t \tag{6}$$

often called **an integral with variable upper bound limit**.

The Integral and the Derivative



The Integral and the Derivative

Lemma 0.21

If $f \in \mathcal{R}[a, b]$ and the function f is continuous at a point $x \in [a, b]$, then the function F defined on $[a, b]$ by Equation(6) is differentiable at the point x, and the following equality holds:

$$F'(x) = f(x)$$

The Integral and the Derivative

Theorem 0.22

Every continuous function $f : [a, b] \to \mathbb{R}$ *on the closed interval* $[a, b]$ *has a primitive, and every primitive of* f *on* $[a, b]$ *has the form*

$$\mathcal{F}(x) = \int_a^x f(t)\,\mathrm{d}t + C$$

Definition 0.23

A continuous function $x \to F(x)$ on an interval of the real line is called a primitive (or generalized primitive) of the function $x \to f(x)$ defined on the same interval if the relation $\mathcal{F}'(x) = f(x)$ holds at all points of the interval, with only a finite number of exceptions.

The Newton-Leibniz Formula

Theorem 0.24

If $f : [a, b] \to \mathbb{R}$ is a bounded function with a finite number of points of discontinuity, then $f \in \mathcal{R}[a, b]$ and

$$\int_a^b f(x)\,\mathrm{d}x = \mathcal{F}(b) - \mathcal{F}(a) \tag{7}$$

where $\mathcal{F} : [a, b] \to \mathbb{R}$ is any primitive of f on $[a, b]$.

Example 0.25

Compute the definite integrals below:

1. $\int_a^b x^n \, \mathrm{d}x$
2. $\int_a^b e^x \, \mathrm{d}x$
3. $\int_0^2 x\sqrt{4 - x^2} \, \mathrm{d}x$

Example 0.26

Compute the limit of the sum using definite integral:

$$\lim_{n\to+\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}\right)$$

Integration by Parts

Theorem 0.27

If the function $u(x)$ and $v(x)$ are continuously differentiable on a closed interval with endpoints a and b, then

$$\int_a^b (u(x)v'(x))\,\mathrm{d}x = (uv)|_a^b - \int_a^b (v(x)u'(x))\,\mathrm{d}x$$

or

$$\int_a^b u(x)\,\mathrm{d}v(x) = (uv)|_a^b - \int_a^b v(x)\,\mathrm{d}u(x)$$

Theorem 0.28

If the function $t \to f(t)$ has continuous derivatives up to order n inclusive on the closed interval with endpoints a and x, then Taylor's formula holds:

$$f(x) = f(a) + \frac{1}{1!} f'(a)(x-a) + \cdots + \frac{1}{(n-1)!} f^{(n-1)}(a)(x-a)^{n-1} + r_{n-1}(a;x),$$

with remainder term

$r_{n-1}(a;x) = r_{n-1}(a;x) = \frac{1}{(n-1)!} \int_a^x f^{(n)}(t)(x-t)^{n-1}\,\mathrm{d}t.$

How to obtain Taylor and Cauchy remainders?

Integration by Parts

1. The Cauchy's formular for the remainder term is

$$r_n(x_0; x) = \frac{1}{n!} f^{(n+1)}(\xi)(x-\xi)^n(x-x_0)$$

2. The Lagrange's formular for the remainder term is

$$r_n(x_0; x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi)(x-x_0)^{n+1}$$

Integration by Parts

Integration by Parts

Example 0.29

Compute the definite integrals below:

1. $\int_1^e x^2 \ln x \, \mathrm{d}x$
2. $\int_0^{\frac{\pi}{2}} \sin^n x \, \mathrm{d}x$
3. $\int_0^{\frac{\pi}{2}} \cos^n x \, \mathrm{d}x$

Integration by Parts

Wallis formular

$$\frac{\pi}{2} = \lim_{n\to\infty} \left[\frac{(2m)!!}{(2m-1)!!}\right] \cdot \frac{1}{2n+1}$$

hint: $\int_0^{\frac{\pi}{2}} \sin^{2m+1} x \, \mathrm{d}x \le \int_0^{\frac{\pi}{2}} \sin^{2m} x \, \mathrm{d}x \le \int_0^{\frac{\pi}{2}} \sin^{2m-1} x \, \mathrm{d}x$

Integration by Parts

Integration by Changing Variable

Theorem 0.30

If $\varphi : [\alpha, \beta] \to [a, b]$ *is a continuously differentiable mapping of the closed interval* $[\alpha, \beta]$ *into the closed interval* $[a, b]$ *such that* $\varphi(\alpha) = a$ *and* $\varphi(\beta) = b$*, then for any continuous function* $f(x)$ *on* $[a, b]$ *the function* $f(\varphi(t))\varphi'(t)$ *is continuous on the closed interval* $[\alpha, \beta]$*, and*

$$\int_a^b f(x)\,\mathrm{d}x = \int_\alpha^\beta f(\varphi(t))\varphi'(t)\,\mathrm{d}t. \tag{8}$$

Example 0.31

$$\int_{-1}^{1} \sqrt{1 - x^2}\,\mathrm{d}x$$

Example 0.32

$$\int_{-\pi}^{\pi} \sin mx \cos nx \,\mathrm{d}x = 0, \int_{-\pi}^{\pi} \sin^2 mx \,\mathrm{d}x = \pi, \int_{-\pi}^{\pi} \cos^2 mx \,\mathrm{d}x = \pi, m, n \in \mathbb{N}.$$

$$\int_{-\pi}^{\pi} \sin mx \,\mathrm{d}x = \int_{-\pi}^{\pi} \cos nx \,\mathrm{d}x = 0, n \neq 0$$

Integration by Changing Variable

Example 0.33

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}\,\mathrm{d}x$$

Example 0.34

Let $f \in \mathcal{R}[-a, a]$, show that

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = \begin{cases} 2\int_0^a f(x)\,\mathrm{d}x, & f(-x) = f(x) \\ 0, & f(-x) = -f(x) \end{cases}$$

Example 0.35

Let f be a real function defined on the entire real line $\mathbb{R}$ and having period T, that is $f(x+T) = f(x)$ for all $x \in \mathbb{R}$. If f is integrable on each finite closed interval, then for any $a \in \mathbb{R}$, we have

$$\int_a^{a+T} f(x)\,\mathrm{d}x = \int_0^T f(x)\,\mathrm{d}x.$$

Example 0.36

The quality $\mu = \int_a^b f(x)\,\mathrm{d}x$ is called the integral average value of the function on the closed interval $[a, b]$.
Let $f(x)$ be a function that is defined on $\mathbb{R}$ and integrable on any closed interval. We use f to construct the new function

$$F_\delta(x) = \frac{1}{2\delta}\int_{x-\delta}^{x+\delta} f(t)\,\mathrm{d}t$$

whose value at the point x is the integral average value f in the $\delta-$neighbohood of x.
We shall show that $F_\delta(x)$ is, compared to f, more regular. More precisely, if f is integrable on any close interval $[a, b]$, the $F_\delta(x)$ is continuous on $\mathbb{R}$, and if $f \in C(\mathbb{R})$, then $F_\delta(x) \in C^{(1)}(\mathbb{R})$.

Integration by Changing Variable

Integration by Changing Variable

The last slide!