Chapter 5

Sequences and Series of Functions

In this chapter, we define and study the convergence of sequences and series of functions. There are many different ways to define the convergence of a sequence of functions, and different definitions lead to inequivalent types of convergence. We consider here two basic types: pointwise and uniform convergence.

5.1. Pointwise convergence

Pointwise convergence defines the convergence of functions in terms of the convergence of their values at each point of their domain.

Definition 5.1. Suppose that (f_n) is a sequence of functions $f_n : A \to \mathbb{R}$ and $f : A \to \mathbb{R}$. Then $f_n \to f$ pointwise on A if $f_n(x) \to f(x)$ as $n \to \infty$ for every $x \in A$.

We say that the sequence (f_n) converges pointwise if it converges pointwise to some function f, in which case

$$f(x) = \lim_{n\to\infty} f_n(x).$$

Pointwise convergence is, perhaps, the most natural way to define the convergence of functions, and it is one of the most important. Nevertheless, as the following examples illustrate, it is not as well-behaved as one might initially expect.

Example 5.2. Suppose that $f_n : (0, 1) \to \mathbb{R}$ is defined by

$$f_n(x) = \frac{n}{nx + 1}.$$

Then, since $x \neq 0$,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{x + 1/n} = \frac{1}{x},$$

so $f_n \to f$ pointwise where $f : (0,1) \to \mathbb{R}$ is given by

$$f(x) = \frac{1}{x}.$$

We have $|f_n(x)| < n$ for all $x \in (0,1)$, so each f_n is bounded on $(0,1)$, but their pointwise limit f is not. Thus, pointwise convergence does not, in general, preserve boundedness.

Example 5.3. Suppose that $f_n : [0,1] \to \mathbb{R}$ is defined by $f_n(x) = x^n$. If $0 \leq x < 1$, then $x^n \to 0$ as $n \to \infty$, while if $x = 1$, then $x^n \to 1$ as $n \to \infty$. So $f_n \to f$ pointwise where

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

Although each f_n is continuous on $[0,1]$, their pointwise limit f is not (it is discontinuous at 1). Thus, pointwise convergence does not, in general, preserve continuity.

Example 5.4. Define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 2n^2 x & \text{if } 0 \leq x \leq 1/(2n) \\ 2n^2(1/n - x) & \text{if } 1/(2n) < x < 1/n, \\ 0 & 1/n \leq x \leq 1. \end{cases}$$

If $0 < x \leq 1$, then $f_n(x) = 0$ for all $n \geq 1/x$, so $f_n(x) \to 0$ as $n \to \infty$; and if $x = 0$, then $f_n(x) = 0$ for all n, so $f_n(x) \to 0$ also. It follows that $f_n \to 0$ pointwise on $[0,1]$. This is the case even though $\max f_n = n \to \infty$ as $n \to \infty$. Thus, a pointwise convergent sequence of functions need not be bounded, even if it converges to zero.

Example 5.5. Define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \frac{\sin nx}{n}.$$

Then $f_n \to 0$ pointwise on $\mathbb{R}$. The sequence (f_n') of derivatives $f_n'(x) = \cos nx$ does not converge pointwise on $\mathbb{R}$; for example,

$$f_n'(\pi) = (-1)^n$$

does not converge as $n \to \infty$. Thus, in general, one cannot differentiate a pointwise convergent sequence. This is because the derivative of a small, rapidly oscillating function may be large.

Example 5.6. Define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \frac{x^2}{\sqrt{x^2 + 1/n}}.$$

If $x \neq 0$, then

$$\lim_{n\to\infty} \frac{x^2}{\sqrt{x^2 + 1/n}} = \frac{x^2}{|x|} = |x|$$

while $f_n(0) = 0$ for all $n \in \mathbb{N}$, so $f_n \to |x|$ pointwise on $\mathbb{R}$. The limit $|x|$ not differentiable at 0 even though all of the f_n are differentiable on $\mathbb{R}$. (The f_n "round off" the corner in the absolute value function.)

Example 5.7. Define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \left(1 + \frac{x}{n}\right)^n.$$

Then by the limit formula for the exponential, which we do not prove here, $f_n \to e^x$ pointwise on $\mathbb{R}$.

5.2. Uniform convergence

In this section, we introduce a stronger notion of convergence of functions than pointwise convergence, called uniform convergence. The difference between pointwise convergence and uniform convergence is analogous to the difference between continuity and uniform continuity.

Definition 5.8. Suppose that (f_n) is a sequence of functions $f_n : A \to \mathbb{R}$ and $f : A \to \mathbb{R}$. Then $f_n \to f$ uniformly on A if, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$n > N \text{ implies that } |f_n(x) - f(x)| < \epsilon \text{ for all } x \in A.$$

When the domain A of the functions is understood, we will often say $f_n \to f$ uniformly instead of uniformly on A.

The crucial point in this definition is that N depends only on ϵ and not on $x \in A$, whereas for a pointwise convergent sequence N may depend on both ϵ and x. A uniformly convergent sequence is always pointwise convergent (to the same limit), but the converse is not true. If for some $\epsilon > 0$ one needs to choose arbitrarily large N for different $x \in A$, meaning that there are sequences of values which converge arbitrarily slowly on A, then a pointwise convergent sequence of functions is not uniformly convergent.

Example 5.9. The sequence $f_n(x) = x^n$ in Example 5.3 converges pointwise on $[0, 1]$ but not uniformly on $[0, 1]$. For $0 \le x < 1$ and $0 < \epsilon < 1$, we have

$$|f_n(x) - f(x)| = |x^n| < \epsilon$$

if and only if $0 \le x < \epsilon^{1/n}$. Since $\epsilon^{1/n} < 1$ for all $n \in \mathbb{N}$, no N works for all x sufficiently close to 1 (although there is no difficulty at $x = 1$). The sequence does, however, converge uniformly on $[0, b]$ for every $0 \le b < 1$; for $0 < \epsilon < 1$, we can take $N = \log \epsilon / \log b$.

Example 5.10. The pointwise convergent sequence in Example 5.4 does not converge uniformly. If it did, it would have to converge to the pointwise limit 0, but

$$\left| f_n\left(\frac{1}{2n}\right) \right| = n,$$

so for no $\epsilon > 0$ does there exist an $N \in \mathbb{N}$ such that $|f_n(x) - 0| < \epsilon$ for all $x \in A$ and $n > N$, since this inequality fails for $n \ge \epsilon$ if $x = 1/(2n)$.

Example 5.11. The functions in Example 5.5 converge uniformly to 0 on $\mathbb{R}$, since

$$|f_n(x)| = \frac{|\sin nx|}{n} \le \frac{1}{n},$$

so $|f_n(x) - 0| < \epsilon$ for all $x \in \mathbb{R}$ if $n > 1/\epsilon$.

5.3. Cauchy condition for uniform convergence

The Cauchy condition in Definition 1.9 provides a necessary and sufficient condition for a sequence of real numbers to converge. There is an analogous uniform Cauchy condition that provides a necessary and sufficient condition for a sequence of functions to converge uniformly.

Definition 5.12. A sequence (f_n) of functions $f_n : A \to \mathbb{R}$ is uniformly Cauchy on A if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$m, n > N \text{ implies that } |f_m(x) - f_n(x)| < \epsilon \text{ for all } x \in A.$$

The key part of the following proof is the argument to show that a pointwise convergent, uniformly Cauchy sequence converges uniformly.

Theorem 5.13. A sequence (f_n) of functions $f_n : A \to \mathbb{R}$ converges uniformly on A if and only if it is uniformly Cauchy on A.

Proof. Suppose that (f_n) converges uniformly to f on A. Then, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \frac{\epsilon}{2} \qquad \text{for all } x \in A \text{ if } n > N.$$

It follows that if $m, n > N$ then

$$|f_m(x) - f_n(x)| \le |f_m(x) - f(x)| + |f(x) - f_n(x)| < \epsilon \qquad \text{for all } x \in A,$$

which shows that (f_n) is uniformly Cauchy.

Conversely, suppose that (f_n) is uniformly Cauchy. Then for each $x \in A$, the real sequence $(f_n(x))$ is Cauchy, so it converges by the completeness of $\mathbb{R}$. We define $f : A \to \mathbb{R}$ by

$$f(x) = \lim_{n\to\infty} f_n(x),$$

and then $f_n \to f$ pointwise.

To prove that $f_n \to f$ uniformly, let $\epsilon > 0$. Since (f_n) is uniformly Cauchy, we can choose $N \in \mathbb{N}$ (depending only on ϵ) such that

$$|f_m(x) - f_n(x)| < \frac{\epsilon}{2} \qquad \text{for all } x \in A \text{ if } m, n > N.$$

Let $n > N$ and $x \in A$. Then for every $m > N$ we have

$$|f_n(x) - f(x)| \le |f_n(x) - f_m(x)| + |f_m(x) - f(x)| < \frac{\epsilon}{2} + |f_m(x) - f(x)|.$$

Since $f_m(x) \to f(x)$ as $m \to \infty$, we can choose $m > N$ (depending on x, but it doesn't matter since m doesn't appear in the final result) such that

$$|f_m(x) - f(x)| < \frac{\epsilon}{2}.$$

It follows that if $n > N$, then

$$|f_n(x) - f(x)| < \epsilon \qquad \text{for all } x \in A,$$

which proves that $f_n \to f$ uniformly.

Alternatively, we can take the limit as $m \to \infty$ in the Cauchy condition to get for all $x \in A$ and $n > N$ that

$$|f(x) - f_n(x)| = \lim_{m\to\infty} |f_m(x) - f_n(x)| \leq \frac{\epsilon}{2} < \epsilon.$$

□

5.4. Properties of uniform convergence

In this section we prove that, unlike pointwise convergence, uniform convergence preserves boundedness and continuity. Uniform convergence does not preserve differentiability any better than pointwise convergence. Nevertheless, we give a result that allows us to differentiate a convergent sequence; the key assumption is that the derivatives converge uniformly.

5.4.1. Boundedness. First, we consider the uniform convergence of bounded functions.

Theorem 5.14. Suppose that $f_n : A \to \mathbb{R}$ is bounded on A for every $n \in \mathbb{N}$ and $f_n \to f$ uniformly on A. Then $f : A \to \mathbb{R}$ is bounded on A.

Proof. Taking $\epsilon = 1$ in the definition of the uniform convergence, we find that there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < 1 \qquad \text{for all } x \in A \text{ if } n > N.$$

Choose some $n > N$. Then, since f_n is bounded, there is a constant $M_n \geq 0$ such that

$$|f_n(x)| \leq M_n \qquad \text{for all } x \in A.$$

It follows that

$$|f(x)| \leq |f(x) - f_n(x)| + |f_n(x)| < 1 + M_n \qquad \text{for all } x \in A,$$

meaning that f is bounded on A (by $1 + M_n$). □

We do not assume here that all the functions in the sequence are bounded by the same constant. (If they were, the pointwise limit would also be bounded by that constant.) In particular, it follows that if a sequence of bounded functions converges pointwise to an unbounded function, then the convergence is not uniform.

Example 5.15. The sequence of functions $f_n : (0, 1) \to \mathbb{R}$ in Example 5.2, defined by

$$f_n(x) = \frac{n}{nx + 1},$$

cannot converge uniformly on $(0, 1)$, since each f_n is bounded on $(0, 1)$, but their pointwise limit $f(x) = 1/x$ is not. The sequence (f_n) does, however, converge uniformly to f on every interval $[a, 1)$ with $0 < a < 1$. To prove this, we estimate for $a \leq x < 1$ that

$$|f_n(x) - f(x)| = \left|\frac{n}{nx + 1} - \frac{1}{x}\right| = \frac{1}{x(nx + 1)} < \frac{1}{nx^2} \leq \frac{1}{na^2}.$$

Thus, given $\epsilon > 0$ choose $N = 1/(a^2\epsilon)$, and then

$$|f_n(x) - f(x)| < \epsilon \qquad \text{for all } x \in [a, 1) \text{ if } n > N,$$

which proves that $f_n \to f$ uniformly on $[a,1)$. Note that

$$|f(x)| \leq \frac{1}{a} \qquad \text{for all } x \in [a,1)$$

so the uniform limit f is bounded on $[a,1)$, as Theorem 5.14 requires.

5.4.2. Continuity. One of the most important property of uniform convergence is that it preserves continuity. We use an "$\epsilon/3$" argument to get the continuity of the uniform limit f from the continuity of the f_n.

Theorem 5.16. If a sequence (f_n) of continuous functions $f_n : A \to \mathbb{R}$ converges uniformly on $A \subset \mathbb{R}$ to $f : A \to \mathbb{R}$, then f is continuous on A.

Proof. Suppose that $c \in A$ and $\epsilon > 0$ is given. Then, for every $n \in \mathbb{N}$,

$$|f(x) - f(c)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(c)| + |f_n(c) - f(c)|.$$

By the uniform convergence of (f_n), we can choose $n \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \frac{\epsilon}{3} \qquad \text{for all } x \in A,$$

and for such an n it follows that

$$|f(x) - f(c)| < |f_n(x) - f_n(c)| + \frac{2\epsilon}{3}.$$

(Here we use the fact that f_n is close to f at both x and c, where x is an an arbitrary point in a neighborhood of c; this is where we use the uniform convergence in a crucial way.)

Since f_n is continuous on A, there exists $\delta > 0$ such that

$$|f_n(x) - f_n(c)| < \frac{\epsilon}{3} \qquad \text{if } |x - c| < \delta \text{ and } x \in A,$$

which implies that

$$|f(x) - f(c)| < \epsilon \qquad \text{if } |x - c| < \delta \text{ and } x \in A.$$

This proves that f is continuous. □

This result can be interpreted as justifying an "exchange in the order of limits"

$$\lim_{n\to\infty} \lim_{x\to c} f_n(x) = \lim_{x\to c} \lim_{n\to\infty} f_n(x).$$

Such exchanges of limits always require some sort of condition for their validity — in this case, the uniform convergence of f_n to f is sufficient, but pointwise convergence is not.

It follows from Theorem 5.16 that if a sequence of continuous functions converges pointwise to a discontinuous function, as in Example 5.3, then the convergence is not uniform. The converse is not true, however, and the pointwise limit of a sequence of continuous functions may be continuous even if the convergence is not uniform, as in Example 5.4.

5.4.3. Differentiability. The uniform convergence of differentiable functions does not, in general, imply anything about the convergence of their derivatives or the differentiability of their limit. As noted above, this is because the values of two functions may be close together while the values of their derivatives are far apart (if, for example, one function varies slowly while the other oscillates rapidly, as in Example 5.5). Thus, we have to impose strong conditions on a sequence of functions and their derivatives if we hope to prove that $f_n \to f$ implies $f_n' \to f'$.

The following example shows that the limit of the derivatives need not equal the derivative of the limit even if a sequence of differentiable functions converges uniformly and their derivatives converge pointwise.

Example 5.17. Consider the sequence (f_n) of functions $f_n : \mathbb{R} \to \mathbb{R}$ defined by

$$f_n(x) = \frac{x}{1+nx^2}.$$

Then $f_n \to 0$ uniformly on $\mathbb{R}$. To see this, we write

$$|f_n(x)| = \frac{1}{\sqrt{n}}\left(\frac{\sqrt{n}|x|}{1+nx^2}\right) = \frac{1}{\sqrt{n}}\left(\frac{t}{1+t^2}\right)$$

where $t = \sqrt{n}|x|$. We have

$$\frac{t}{1+t^2} \le \frac{1}{2} \qquad \text{for all } t \in \mathbb{R},$$

since $(1-t)^2 \ge 0$, which implies that $2t \le 1+t^2$. Using this inequality, we get

$$|f_n(x)| \le \frac{1}{2\sqrt{n}} \qquad \text{for all } x \in \mathbb{R}.$$

Hence, given $\epsilon > 0$, choose $N = 1/(4\epsilon^2)$. Then

$$|f_n(x)| < \epsilon \qquad \text{for all } x \in \mathbb{R} \text{ if } n > N,$$

which proves that (f_n) converges uniformly to 0 on $\mathbb{R}$. (Alternatively, we could get the same result by using calculus to compute the maximum value of $|f_n|$ on $\mathbb{R}$.)

Each f_n is differentiable with

$$f_n'(x) = \frac{1-nx^2}{(1+nx^2)^2}.$$

It follows that $f_n' \to g$ pointwise as $n \to \infty$ where

$$g(x) = \begin{cases} 0 & \text{if } x \ne 0, \\ 1 & \text{if } x = 0. \end{cases}$$

The convergence is not uniform since g is discontinuous at 0. Thus, $f_n \to 0$ uniformly, but $f_n'(0) \to 1$, so the limit of the derivatives is not the derivative of the limit.

However, we do get a useful result if we strengthen the assumptions and require that the derivatives converge uniformly, not just pointwise. The proof involves a slightly tricky application of the mean value theorem.

Theorem 5.18. *Suppose that (f_n) is a sequence of differentiable functions $f_n : (a,b) \to \mathbb{R}$ such that $f_n \to f$ pointwise and $f_n' \to g$ uniformly for some $f, g : (a,b) \to \mathbb{R}$. Then f is differentiable on (a,b) and $f' = g$.*

Proof. Let $c \in (a,b)$, and let $\epsilon > 0$ be given. To prove that $f'(c) = g(c)$, we estimate the difference quotient of f in terms of the difference quotients of the f_n:

$$\begin{aligned}\left|\frac{f(x)-f(c)}{x-c} - g(c)\right| \le & \left|\frac{f(x)-f(c)}{x-c} - \frac{f_n(x)-f_n(c)}{x-c}\right| \\ & + \left|\frac{f_n(x)-f_n(c)}{x-c} - f_n'(c)\right| + |f_n'(c) - g(c)|\end{aligned}$$

where $x \in (a,b)$ and $x \neq c$. We want to make each of the terms on the right-hand side of the inequality less than $\epsilon/3$. This is straightforward for the second term (since f_n is differentiable) and the third term (since $f_n' \to g$). To estimate the first term, we approximate f by f_m, use the mean value theorem, and let $m \to \infty$.

Since $f_m - f_n$ is differentiable, the mean value theorem implies that there exists ξ between c and x such that

$$\begin{aligned}\frac{f_m(x)-f_m(c)}{x-c} - \frac{f_n(x)-f_n(c)}{x-c} &= \frac{(f_m-f_n)(x) - (f_m-f_n)(c)}{x-c} \\ &= f_m'(\xi) - f_n'(\xi).\end{aligned}$$

Since (f_n') converges uniformly, it is uniformly Cauchy by Theorem 5.13. Therefore there exists $N_1 \in \mathbb{N}$ such that

$$|f_m'(\xi) - f_n'(\xi)| < \frac{\epsilon}{3} \qquad \text{for all } \xi \in (a,b) \text{ if } m, n > N_1,$$

which implies that

$$\left|\frac{f_m(x)-f_m(c)}{x-c} - \frac{f_n(x)-f_n(c)}{x-c}\right| < \frac{\epsilon}{3}.$$

Taking the limit of this equation as $m \to \infty$, and using the pointwise convergence of (f_m) to f, we get that

$$\left|\frac{f(x)-f(c)}{x-c} - \frac{f_n(x)-f_n(c)}{x-c}\right| \le \frac{\epsilon}{3} \qquad \text{for } n > N_1.$$

Next, since (f_n') converges to g, there exists $N_2 \in \mathbb{N}$ such that

$$|f_n'(c) - g(c)| < \frac{\epsilon}{3} \qquad \text{for all } n > N_2.$$

Choose some $n > \max(N_1, N_2)$. Then the differentiability of f_n implies that there exists $\delta > 0$ such that

$$\left|\frac{f_n(x)-f_n(c)}{x-c} - f_n'(c)\right| < \frac{\epsilon}{3} \qquad \text{if } 0 < |x-c| < \delta.$$

Putting these inequalities together, we get that

$$\left|\frac{f(x)-f(c)}{x-c} - g(c)\right| < \epsilon \qquad \text{if } 0 < |x-c| < \delta,$$

which proves that f is differentiable at c with $f'(c) = g(c)$. □

Like Theorem 5.16, Theorem 5.18 can be interpreted as giving sufficient conditions for an exchange in the order of limits:

$$\lim_{n\to\infty} \lim_{x\to c} \left[\frac{f_n(x)-f_n(c)}{x-c}\right] = \lim_{x\to c} \lim_{n\to\infty} \left[\frac{f_n(x)-f_n(c)}{x-c}\right].$$

It is worth noting that in Theorem 5.18 the derivatives f_n' are not assumed to be continuous. If they are continuous, one can use Riemann integration and the fundamental theorem of calculus (FTC) to give a simpler proof of the theorem, as follows. Fix some $x_0 \in (a, b)$. The uniform convergence $f_n' \to g$ implies that

$$\int_{x_0}^{x} f_n' dx \to \int_{x_0}^{x} g\, dx.$$

(This is the main point: although we cannot differentiate a uniformly convergent sequence, we can integrate it.) It then follows from one direction of the FTC that

$$f_n(x) - f_n(x_0) \to \int_{x_0}^{x} g\, dx,$$

and the pointwise convergence $f_n \to f$ implies that

$$f(x) = f(x_0) + \int_{x_0}^{x} g\, dx.$$

The other direction of the FTC then implies that f is differentiable and $f' = g$.

5.5. Series

The convergence of a series is defined in terms of the convergence of its sequence of partial sums, and any result about sequences is easily translated into a corresponding result about series.

Definition 5.19. Suppose that (f_n) is a sequence of functions $f_n : A \to \mathbb{R}$, and define a sequence (S_n) of partial sums $S_n : A \to \mathbb{R}$ by

$$S_n(x) = \sum_{k=1}^{n} f_k(x).$$

Then the series

$$S(x) = \sum_{n=1}^{\infty} f_n(x)$$

converges pointwise to $S : A \to \mathbb{R}$ on A if $S_n \to S$ as $n \to \infty$ pointwise on A, and uniformly to S on A if $S_n \to S$ uniformly on A.

We illustrate the definition with a series whose partial sums we can compute explicitly.

Example 5.20. The geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

has partial sums

$$S_n(x) = \sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}.$$

Thus, $S_n(x) \to 1/(1-x)$ as $n \to \infty$ if $|x| < 1$ and diverges if $|x| \geq 1$, meaning that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad \text{pointwise on } (-1, 1).$$

Since $1/(1-x)$ is unbounded on $(-1, 1)$, Theorem 5.14 implies that the convergence cannot be uniform.

The series does, however, converges uniformly on $[-\rho, \rho]$ for every $0 \le \rho < 1$. To prove this, we estimate for $|x| \le \rho$ that

$$\left| S_n(x) - \frac{1}{1-x} \right| = \frac{|x|^{n+1}}{1-x} \le \frac{\rho^{n+1}}{1-\rho}.$$

Since $\rho^{n+1}/(1-\rho) \to 0$ as $n \to \infty$, given any $\epsilon > 0$ there exists $N \in \mathbb{N}$, depending only on ϵ and ρ, such that

$$0 \le \frac{\rho^{n+1}}{1-\rho} < \epsilon \qquad \text{for all } n > N.$$

It follows that

$$\left| \sum_{k=0}^{n} x^k - \frac{1}{1-x} \right| < \epsilon \qquad \text{for all } x \in [-\rho, \rho] \text{ and all } n > N,$$

which proves that the series converges uniformly on $[-\rho, \rho]$.

The Cauchy condition for the uniform convergence of sequences immediately gives a corresponding Cauchy condition for the uniform convergence of series.

Theorem 5.21. Let (f_n) be a sequence of functions $f_n : A \to \mathbb{R}$. The series

$$\sum_{n=1}^{\infty} f_n$$

converges uniformly on A if and only if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\left| \sum_{k=m+1}^{n} f_k(x) \right| < \epsilon \qquad \text{for all } x \in A \text{ and all } n > m > N.$$

Proof. Let

$$S_n(x) = \sum_{k=1}^{n} f_k(x) = f_1(x) + f_2(x) + \cdots + f_n(x).$$

From Theorem 5.13 the sequence (S_n), and therefore the series $\sum f_n$, converges uniformly if and only if for every $\epsilon > 0$ there exists N such that

$$|S_n(x) - S_m(x)| < \epsilon \qquad \text{for all } x \in A \text{ and all } n, m > N.$$

Assuming $n > m$ without loss of generality, we have

$$S_n(x) - S_m(x) = f_{m+1}(x) + f_{m+2}(x) + \cdots + f_n(x) = \sum_{k=m+1}^{n} f_k(x),$$

so the result follows. □

This condition says that the sum of any number of consecutive terms in the series gets arbitrarily small sufficiently far down the series.

5.6. The Weierstrass M-test

The following simple criterion for the uniform convergence of a series is very useful. The name comes from the letter traditionally used to denote the constants, or "majorants," that bound the functions in the series.

Theorem 5.22 (Weierstrass M-test)**.** Let (f_n) be a sequence of functions $f_n : A \to \mathbb{R}$, and suppose that for every $n \in \mathbb{N}$ there exists a constant $M_n \geq 0$ such that

$$|f_n(x)| \leq M_n \quad \text{for all } x \in A, \qquad \sum_{n=1}^{\infty} M_n < \infty.$$

Then

$$\sum_{n=1}^{\infty} f_n(x).$$

converges uniformly on A.

Proof. The result follows immediately from the observation that $\sum f_n$ is uniformly Cauchy if $\sum M_n$ is Cauchy.

In detail, let $\epsilon > 0$ be given. The Cauchy condition for the convergence of a real series implies that there exists $N \in \mathbb{N}$ such that

$$\sum_{k=m+1}^{n} M_k < \epsilon \qquad \text{for all } n > m > N.$$

Then for all $x \in A$ and all $n > m > N$, we have

$$\begin{aligned} \left| \sum_{k=m+1}^{n} f_k(x) \right| &\leq \sum_{k=m+1}^{n} |f_k(x)| \\ &\leq \sum_{k=m+1}^{n} M_k \\ &< \epsilon. \end{aligned}$$

Thus, $\sum f_n$ satisfies the uniform Cauchy condition in Theorem 5.21, so it converges uniformly. □

This proof illustrates the value of the Cauchy condition: we can prove the convergence of the series without having to know what its sum is.

Example 5.23. Returning to Example 5.20, we consider the geometric series

$$\sum_{n=0}^{\infty} x^n.$$

If $|x| \leq \rho$ where $0 \leq \rho < 1$, then

$$|x^n| \leq \rho^n, \qquad \sum_{n=0}^{\infty} \rho^n < 1.$$

The M-test, with $M_n = \rho^n$, implies that the series converges uniformly on $[-\rho, \rho]$.

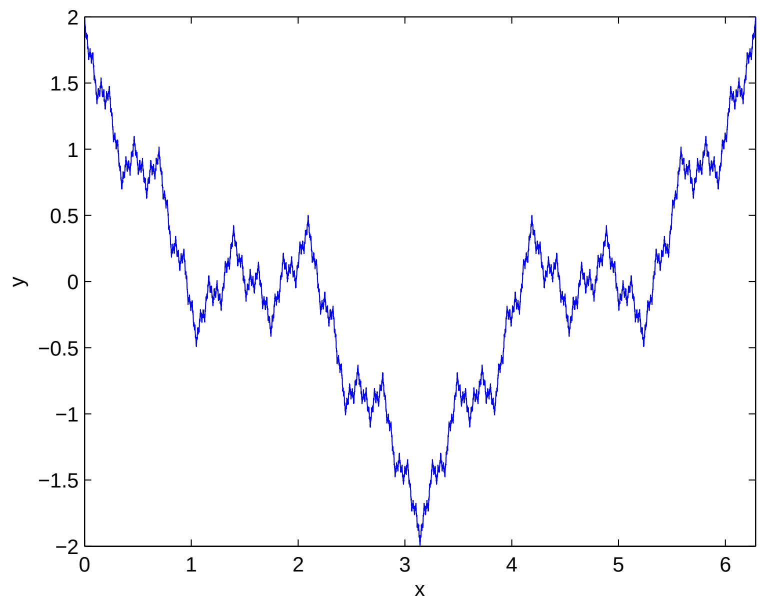

Figure 1. Graph of the Weierstrass continuous, nowhere differentiable function $y = \sum_{n=0}^{\infty} 2^{-n} \cos(3^n x)$ on one period $[0, 2\pi]$.

Example 5.24. The series

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cos\left(3^n x\right)$$

converges uniformly on $\mathbb{R}$ by the M-test since

$$\left|\frac{1}{2^n} \cos\left(3^n x\right)\right| \leq \frac{1}{2^n}, \qquad \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

It then follows from Theorem 5.16 that f is continuous on $\mathbb{R}$. (See Figure 1.)

Taking the formal term-by-term derivative of the series for f, we get a series whose coefficients grow with n,

$$-\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n \sin\left(3^n x\right),$$

so we might expect that there are difficulties in differentiating f. As Figure 2 illustrates, the function does not appear to be smooth at all length-scales. Weierstrass (1872) proved that f is not differentiable at any point of $\mathbb{R}$. Bolzano (1830) had also constructed a continuous, nowhere differentiable function, but his results weren't published until 1922. Subsequently, Tagaki (1903) constructed a similar function to the Weierstrass function whose nowhere-differentiability is easier to prove. Such functions were considered to be highly counter-intuitive and pathological at the time Weierstrass discovered them, and they weren't well-received by many prominent mathematicians.

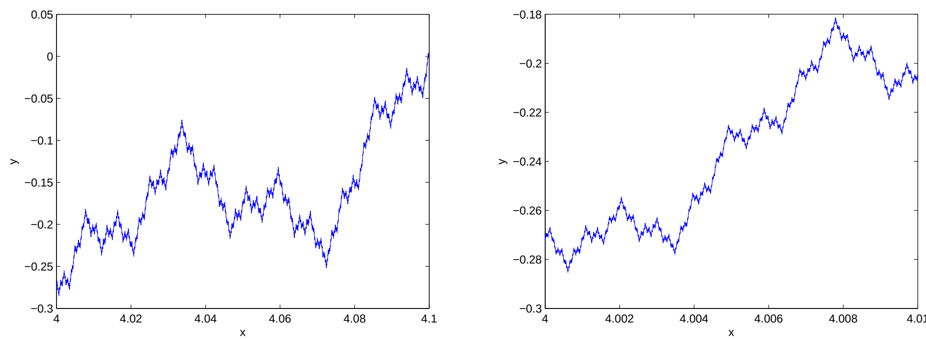

Figure 2. Details of the Weierstrass function showing its self-similar, fractal behavior under rescalings.

If the Weierstrass M-test applies to a series of functions to prove uniform convergence, it also implies that the series converges absolutely, meaning that

$$\sum_{n=1}^{\infty} |f_n(x)| < \infty \qquad \text{for every } x \in A.$$

Thus, the M-test is not applicable to series that converge uniformly but not absolutely.

Absolute convergence of a series is completely different from uniform convergence, and the two concepts should not be confused. Absolute convergence on A is a pointwise condition for each $x \in A$, while uniform convergence is a global condition that involves all points $x \in A$ simultaneously. We illustrate the difference with a rather trivial example.

Example 5.25. Let $f_n : \mathbb{R} \to \mathbb{R}$ be the constant function

$$f_n(x) = \frac{(-1)^{n+1}}{n}.$$

Then $\sum f_n$ converges on $\mathbb{R}$ to the constant function $f(x) = c$, where

$$c = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

is the sum of the alternating harmonic series ($c = \log 2$). The convergence of $\sum f_n$ is uniform on $\mathbb{R}$ since the terms in the series do not depend on x, but the convergence is not absolute at any $x \in \mathbb{R}$ since the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges to infinity.

5.7. The sup-norm

An equivalent, and often clearer, way to describe uniform convergence is in terms of the uniform, or sup, norm.

Definition 5.26. Suppose that $f : A \to \mathbb{R}$. The uniform, or sup, norm $\|f\|$ of f on A is

$$\|f\| = \sup_{x\in A} |f(x)|.$$

A function is bounded on A if and only if $\|f\| < \infty$.

Example 5.27. Let $A = (0, 1)$ and define $f, g, h : (0, 1) \to \mathbb{R}$ by

$$f(x) = x^2, \qquad g(x) = x^2 - x, \qquad h(x) = \frac{1}{x}.$$

Then

$$\|f\| = 1, \quad \|g\| = \frac{1}{4}, \qquad \|h\| = \infty.$$

We have the following characterization of uniform convergence.

Definition 5.28. A sequence (f_n) of functions $f_n : A \to \mathbb{R}$ converges uniformly on A to a function $f : A \to \mathbb{R}$ if

$$\lim_{n\to\infty} \|f_n - f\| = 0.$$

Similarly, we can define a uniformly Cauchy sequence in terms of the sup-norm.

Definition 5.29. A sequence (f_n) of functions $f_n : A \to \mathbb{R}$ is uniformly Cauchy on A if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$m, n > N \text{ implies that } \|f_m - f_n\| < \epsilon.$$

Thus, the uniform convergence of a sequence of functions is defined in exactly the same way as the convergence of a sequence of real numbers with the absolute $|\cdot|$ value replaced by the sup-norm $\|\cdot\|$.

5.8. Spaces of continuous functions

Our previous theorems about continuous functions on compact sets can be restated in a more geometrical way using the sup-norm.

Definition 5.30. Let $K \subset \mathbb{R}$ be a compact set. The space $C(K)$ consists of all continuous functions $f : K \to \mathbb{R}$.

Thus, we think of a function f as a point in a function space $C(K)$, just as we think of a real number x as a point in $\mathbb{R}$.

Theorem 5.31. The space $C(K)$ is a vector space with respect to the usual pointwise definitions of scalar multiplication and addition of functions: If $f, g \in C(K)$ and $k \in \mathbb{R}$, then

$$(kf)(x) = kf(x), \qquad (f + g)(x) = f(x) + g(x).$$

This follows from Theorem 3.15, which states that scalar multiples and sums of continuous functions are continuous and therefore belong to $C(K)$. The algebraic vector-space properties of $C(K)$ follow immediately from those of the real numbers.

Definition 5.32. A normed vector space $(X, \|\cdot\|)$ is a vector space X (which we assume to be real) together with a function $\|\cdot\| : X \to \mathbb{R}$, called a norm on X, such that for all $f, g \in X$ and $k \in \mathbb{R}$:

(1) $0 \leq \|f\| < \infty$ and $\|f\| = 0$ if and only if $f = 0$;
(2) $\|kf\| = |k|\|f\|$;
(3) $\|f + g\| \leq \|f\| + \|g\|$.

We think of $\|f\|$ as defining a "length" of the vector $f \in X$ and $\|f - g\|$ as the corresponding "distance" between $f, g \in X$. (There are typically many ways to define a norm on a vector space satisfying Definition 5.32, each leading to a different notion of the distance between vectors.)

The properties in Definition 5.32 are natural one to require of a length: The length of f is 0 if and only if f is the 0-vector; multiplying a vector by k multiplies its length by $|k|$; and the length of the "hypoteneuse" $f + g$ is less than or equal to the sum of the lengths of the "sides" f, g. Because of this last interpretation, property (3) is referred to as the triangle inequality.

It is straightforward to verify that the sup-norm on $C(K)$ has these properties.

Theorem 5.33. The space $C(K)$ with the sup-norm $\|\cdot\| : C(K) \to \mathbb{R}$ given in Definition 5.26 is a normed vector space.

Proof. From Theorem 3.33, the sup-norm of a continuous function $f : K \to \mathbb{R}$ on a compact set K is finite, and it is clearly nonnegative, so $0 \leq \|f\| < \infty$. If $\|f\| = 0$, then $\sup_{x \in K} |f(x)| = 0$, which implies that $f(x) = 0$ for every $x \in K$, meaning that $f = 0$ is the zero function.

We also have

$$\|kf\| = \sup_{x \in K} |k(f(x)| = |k| \sup_{x \in K} |f(x)| = k\|f\|,$$

and

$$\begin{aligned}
\|f + g\| &= \sup_{x \in K} |(f(x) + g(x)| \\
&\leq \sup_{x \in K} \{|f(x)| + |g(x)|\} \\
&\leq \sup_{x \in K} |f(x)| + \sup_{x \in K} |g(x)| \\
&\leq \|f\| + \|g\|,
\end{aligned}$$

which verifies the properties of a norm. □

Definition 5.34. A sequence (f_n) in a normed vector space $(X, \|\cdot\|)$ converges to $f \in X$ if $\|f_n - f\| \to 0$ as $n \to \infty$. That is, if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$n > N \text{ implies that } \|f_n - f\| < \epsilon.$$

The sequence is a Cauchy sequence for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$m, n > N \text{ implies that } \|f_m - f_n\| < \epsilon.$$

Definition 5.35. A normed vector space is complete if every Cauchy sequence converges. A complete normed linear space is called a Banach space.

Theorem 5.36. The space $C(K)$ with the sup-norm is a Banach space.

Proof. The space $C(K)$ with the sup-norm is a normed space from Theorem 5.33. Theorem 5.13 implies that it is complete. $\square$