Chapter 6

# Power Series

Power series are one of the most useful type of series in analysis. For example, we can use them to define transcendental functions such as the exponential and trigonometric functions (and many other less familiar functions).

## 6.1. Introduction

A power series (centered at 0) is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \ldots.$$

where the $a_n$ are some coefficients. If all but finitely many of the $a_n$ are zero, then the power series is a polynomial function, but if infinitely many of the $a_n$ are nonzero, then we need to consider the convergence of the power series.

The basic facts are these: Every power series has a radius of convergence $0 \leq R \leq \infty$, which depends on the coefficients $a_n$. The power series converges absolutely in $|x| < R$ and diverges in $|x| > R$, and the convergence is uniform on every interval $|x| < \rho$ where $0 \leq \rho < R$. If $R > 0$, the sum of the power series is infinitely differentiable in $|x| < R$, and its derivatives are given by differentiating the original power series term-by-term.

Power series work just as well for complex numbers as real numbers, and are in fact best viewed from that perspective, but we restrict our attention here to real-valued power series.

**Definition 6.1.** Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers and $c \in \mathbb{R}$. The power series centered at $c$ with coefficients $a_n$ is the series

$$\sum_{n=0}^{\infty} a_n (x - c)^n.$$

Here are some power series centered at 0:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \dots,$$

$$\sum_{n=0}^{\infty} \frac{1}{n!}x^n = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots,$$

$$\sum_{n=0}^{\infty} (n!)x^n = 1 + x + 2x^2 + 6x^3 + 24x^4 + \dots,$$

$$\sum_{n=0}^{\infty} (-1)^n x^{2^n} = x - x^2 + x^4 - x^8 + \dots;$$

and here is a power series centered at 1:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \dots.$$

The power series in Definition 6.1 is a formal expression, since we have not said anything about its convergence. By changing variables $x \mapsto (x-c)$, we can assume without loss of generality that a power series is centered at 0, and we will do so when it's convenient.

## 6.2. Radius of convergence

First, we prove that every power series has a radius of convergence.

**Theorem 6.2.** Let

$$\sum_{n=0}^{\infty} a_n(x-c)^n$$

be a power series. There is an $0 \le R \le \infty$ such that the series converges absolutely for $0 \le |x-c| < R$ and diverges for $|x-c| > R$. Furthermore, if $0 \le \rho < R$, then the power series converges uniformly on the interval $|x-c| \le \rho$, and the sum of the series is continuous in $|x-c| < R$.

**Proof.** Assume without loss of generality that $c = 0$ (otherwise, replace $x$ by $x-c$). Suppose the power series

$$\sum_{n=0}^{\infty} a_n x_0^n$$

converges for some $x_0 \in \mathbb{R}$ with $x_0 \neq 0$. Then its terms converge to zero, so they are bounded and there exists $M \ge 0$ such that

$$|a_n x_0^n| \le M \qquad \text{for } n = 0, 1, 2, \dots.$$

If $|x| < |x_0|$, then

$$|a_n x^n| = |a_n x_0^n| \left|\frac{x}{x_0}\right|^n \le M r^n, \qquad r = \left|\frac{x}{x_0}\right| < 1.$$

Comparing the power series with the convergent geometric series $\sum M r^n$, we see that $\sum a_n x^n$ is absolutely convergent. Thus, if the power series converges for some $x_0 \in \mathbb{R}$, then it converges absolutely for every $x \in \mathbb{R}$ with $|x| < |x_0|$.

Let

$$R = \sup\left\{|x| \geq 0 : \sum a_n x^n \text{ converges}\right\}.$$

If $R = 0$, then the series converges only for $x = 0$. If $R > 0$, then the series converges absolutely for every $x \in \mathbb{R}$ with $|x| < R$, because it converges for some $x_0 \in \mathbb{R}$ with $|x| < |x_0| < R$. Moreover, the definition of $R$ implies that the series diverges for every $x \in \mathbb{R}$ with $|x| > R$. If $R = \infty$, then the series converges for all $x \in \mathbb{R}$.

Finally, let $0 \leq \rho < R$ and suppose $|x| \leq \rho$. Choose $\sigma > 0$ such that $\rho < \sigma < R$. Then $\sum |a_n \sigma^n|$ converges, so $|a_n \sigma^n| \leq M$, and therefore

$$|a_n x^n| = |a_n \sigma^n| \left|\frac{x}{\sigma}\right|^n \leq |a_n \sigma^n| \left|\frac{\rho}{\sigma}\right|^n \leq M r^n,$$

where $r = \rho/\sigma < 1$. Since $\sum M r^n < \infty$, the $M$-test (Theorem 5.22) implies that the series converges uniformly on $|x| \leq \rho$, and then it follows from Theorem 5.16 that the sum is continuous on $|x| \leq \rho$. Since this holds for every $0 \leq \rho < R$, the sum is continuous in $|x| < R$. □

The following definition therefore makes sense for every power series.

**Definition 6.3.** If the power series

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

converges for $|x - c| < R$ and diverges for $|x - c| > R$, then $0 \leq R \leq \infty$ is called the radius of convergence of the power series.

Theorem 6.2 does not say what happens at the endpoints $x = c \pm R$, and in general the power series may converge or diverge there. We refer to the set of all points where the power series converges as its interval of convergence, which is one of

$$(c - R, c + R), \quad (c - R, c + R], \quad [c - R, c + R), \quad [c - R, c + R].$$

We will not discuss any general theorems about the convergence of power series at the endpoints (e.g. the Abel theorem).

Theorem 6.2 does not give an explicit expression for the radius of convergence of a power series in terms of its coefficients. The ratio test gives a simple, but useful, way to compute the radius of convergence, although it doesn't apply to every power series.

**Theorem 6.4.** Suppose that $a_n \neq 0$ for all sufficiently large $n$ and the limit

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$$

exists or diverges to infinity. Then the power series

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

has radius of convergence $R$.

**Proof.** Let

$$r = \lim_{n\to\infty} \left| \frac{a_{n+1}(x-c)^{n+1}}{a_n(x-c)^n} \right| = |x-c| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

By the ratio test, the power series converges if $0 \le r < 1$, or $|x-c| < R$, and diverges if $1 < r \le \infty$, or $|x-c| > R$, which proves the result. $\square$

The root test gives an expression for the radius of convergence of a general power series.

**Theorem 6.5** (Hadamard)**.** The radius of convergence $R$ of the power series

$$\sum_{n=0}^{\infty} a_n(x-c)^n$$

is given by

$$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}$$

where $R = 0$ if the lim sup diverges to $\infty$, and $R = \infty$ if the lim sup is 0.

**Proof.** Let

$$r = \limsup_{n\to\infty} |a_n(x-c)^n|^{1/n} = |x-c| \limsup_{n\to\infty} |a_n|^{1/n}.$$

By the root test, the series converges if $0 \le r < 1$, or $|x-c| < R$, and diverges if $1 < r \le \infty$, or $|x-c| > R$, which proves the result. $\square$

This theorem provides an alternate proof of Theorem 6.2 from the root test; in fact, our proof of Theorem 6.2 is more-or-less a proof of the root test.

## 6.3. Examples of power series

We consider a number of examples of power series and their radii of convergence.

**Example 6.6.** The geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots$$

has radius of convergence

$$R = \lim_{n\to\infty} \frac{1}{1} = 1.$$

so it converges for $|x| < 1$, to $1/(1-x)$, and diverges for $|x| > 1$. At $x = 1$, the series becomes

$$1 + 1 + 1 + 1 + \dots$$

and at $x = -1$ it becomes

$$1 - 1 + 1 - 1 + 1 - \dots,$$

so the series diverges at both endpoints $x = \pm 1$. Thus, the interval of convergence of the power series is $(-1, 1)$. The series converges uniformly on $[-\rho, \rho]$ for every $0 \le \rho < 1$ but does not converge uniformly on $(-1, 1)$ (see Example 5.20. Note that although the function $1/(1-x)$ is well-defined for all $x \neq 1$, the power series only converges to it when $|x| < 1$.

**Example 6.7.** The series

$$\sum_{n=1}^{\infty} \frac{1}{n} x^n = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \dots$$

has radius of convergence

$$R = \lim_{n\to\infty} \frac{1/n}{1/(n+1)} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1.$$

At $x = 1$, the series becomes the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots,$$

which diverges, and at $x = -1$ it is minus the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \dots,$$

which converges, but not absolutely. Thus the interval of convergence of the power series is $[-1, 1)$. The series converges uniformly on $[-\rho, \rho]$ for every $0 \le \rho < 1$ but does not converge uniformly on $(-1, 1)$.

**Example 6.8.** The power series

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2!}x + \frac{1}{3!}x^3 + \dots$$

has radius of convergence

$$R = \lim_{n\to\infty} \frac{1/n!}{1/(n+1)!} = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} (n+1) = \infty,$$

so it converges for all $x \in \mathbb{R}$. Its sum provides a definition of the exponential function $\exp : \mathbb{R} \to \mathbb{R}$. (See Section 6.5.)

**Example 6.9.** The power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \dots$$

has radius of convergence $R = \infty$, and it converges for all $x \in \mathbb{R}$. Its sum provides a definition of the cosine function $\cos : \mathbb{R} \to \mathbb{R}$.

**Example 6.10.** The series

$$\sum_{n=0\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \dots$$

has radius of convergence $R = \infty$, and it converges for all $x \in \mathbb{R}$. Its sum provides a definition of the sine function $\sin : \mathbb{R} \to \mathbb{R}$.

**Example 6.11.** The power series

$$\sum_{n=0}^{\infty} (n!) x^n = 1 + x + (2!)x + (3!)x^3 + (4!)x^4 + \dots$$

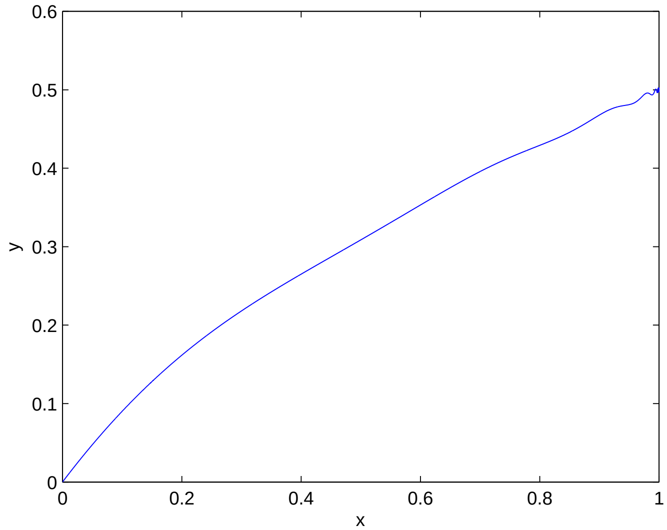


**Figure 1.** Graph of the lacunary power series $y = \sum_{n=0}^{\infty}(-1)^n x^{2^n}$ on $[0, 1)$. It appears relatively well-behaved; however, the small oscillations visible near $x = 1$ are not a numerical artifact.

has radius of convergence

$$R = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0,$$

so it converges only for $x = 0$. If $x \neq 0$, its terms grow larger once $n > 1/|x|$ and $|(n!)x^n| \to \infty$ as $n \to \infty$.

**Example 6.12.** The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \dots$$

has radius of convergence

$$R = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}/n}{(-1)^{n+2}/(n+1)} \right| = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+1/n} = 1,$$

so it converges if $|x-1| < 1$ and diverges if $|x-1| > 1$. At the endpoint $x = 2$, the power series becomes the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots,$$

which converges. At the endpoint $x = 0$, the power series becomes the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots,$$

which diverges. Thus, the interval of convergence is $(0, 2]$.

**Example 6.13.** The power series

$$\sum_{n=0}^{\infty}(-1)^n x^{2^n} = x - x^2 + x^4 - x^8 + x^{16} - x^{32} + \dots$$

with

$$a_n = \begin{cases} 1 & \text{if } n = 2^k, \\ 0 & \text{if } n \neq 2^k, \end{cases}$$

has radius of convergence $R = 1$. To prove this, note that the series converges for $|x| < 1$ by comparison with the convergent geometric series $\sum |x|^n$, since

$$|a_n x^n| = \begin{cases} |x|^n & \text{if } n = 2^k, \\ 0 \leq |x|^n & \text{if } n \neq 2^k. \end{cases}$$

If $|x| > 1$, the terms do not approach 0 as $n \to \infty$, so the series diverges. Alternatively, we have

$$|a_n|^{1/n} = \begin{cases} 1 & \text{if } n = 2^k, \\ 0 & \text{if } n \neq 2^k, \end{cases}$$

so

$$\limsup_{n\to\infty} |a_n|^{1/n} = 1$$

and the root test (Theorem 6.5) gives $R = 1$. The series does not converge at either endpoint $x = \pm 1$, so its interval of convergence is $(-1, 1)$.

There are successively longer gaps (or "lacuna") between the powers with non-zero coefficients. Such series are called lacunary power series, and they have many interesting properties. For example, although the series does not converge at $x = 1$, one can ask if

$$\lim_{x\to 1^-}\left[\sum_{n=0}^{\infty}(-1)^n x^{2^n}\right]$$

exists. In a plot of this sum on $[0, 1)$, shown in Figure 1, the function appears relatively well-behaved near $x = 1$. However, Hardy (1907) proved that the function has infinitely many, very small oscillations as $x \to 1^-$, as illustrated in Figure 2, and the limit does not exist. Subsequent results by Hardy and Littlewood (1926) showed, under suitable assumptions on the growth of the "gaps" between non-zero coefficients, that if the limit of a lacunary power series as $x \to 1^-$ exists, then the series must converge at $x = 1$. Since the lacunary power series considered here does not converge at 1, its limit as $x \to 1^-$ cannot exist

## 6.4. Differentiation of power series

We saw in Section 5.4.3 that, in general, one cannot differentiate a uniformly convergent sequence or series. We can, however, differentiate power series, and they behaves as nicely as one can imagine in this respect. The sum of a power series

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$

is infinitely differentiable inside its interval of convergence, and its derivative

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots$$

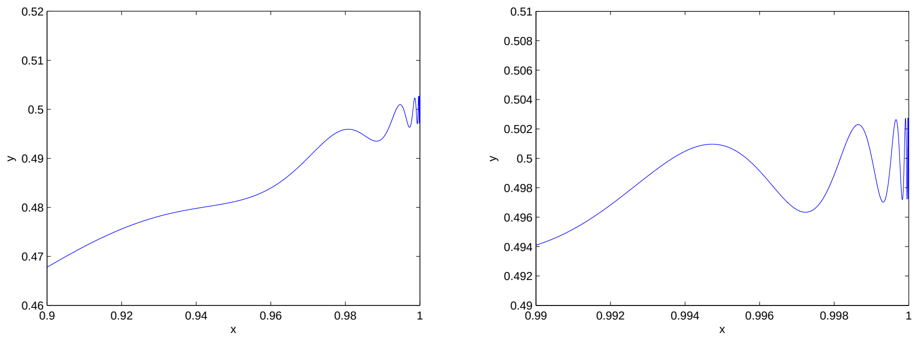


**Figure 2.** Details of the lacunary power series $\sum_{n=0}^{\infty}(-1)^n x^{2^n}$ near $x = 1$, showing its oscillatory behavior and the nonexistence of a limit as $x \to 1^-$.

is given by term-by-term differentiation. To prove this, we first show that the term-by-term derivative of a power series has the same radius of convergence as the original power series. The idea is that the geometrical decay of the terms of the power series inside its radius of convergence dominates the algebraic growth of the factor $n$.

**Theorem 6.14.** Suppose that the power series

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

has radius of convergence $R$. Then the power series

$$\sum_{n=1}^{\infty} n a_n (x-c)^{n-1}$$

also has radius of convergence $R$.

**Proof.** Assume without loss of generality that $c = 0$, and suppose $|x| < R$. Choose $\rho$ such that $|x| < \rho < R$, and let

$$r = \frac{|x|}{\rho}, \qquad 0 < r < 1.$$

To estimate the terms in the differentiated power series by the terms in the original series, we rewrite their absolute values as follows:

$$\left| n a_n x^{n-1} \right| = \frac{n}{\rho} \left( \frac{|x|}{\rho} \right)^{n-1} |a_n \rho^n| = \frac{n r^{n-1}}{\rho} |a_n \rho^n|.$$

The ratio test shows that the series $\sum n r^{n-1}$ converges, since

$$\lim_{n\to\infty} \left[ \frac{(n+1)r^n}{n r^{n-1}} \right] = \lim_{n\to\infty} \left[ \left( 1 + \frac{1}{n} \right) r \right] = r < 1,$$

so the sequence $(n r^{n-1})$ is bounded, by $M$ say. It follows that

$$\left| n a_n x^{n-1} \right| \le \frac{M}{\rho} |a_n \rho^n| \qquad \text{for all } n \in \mathbb{N}.$$

The series $\sum |a_n \rho^n|$ converges, since $\rho < R$, so the comparison test implies that $\sum n a_n x^{n-1}$ converges absolutely.

Conversely, suppose $|x| > R$. Then $\sum |a_n x^n|$ diverges (since $\sum a_n x^n$ diverges) and

$$\left|n a_n x^{n-1}\right| \geq \frac{1}{|x|} |a_n x^n|$$

for $n \geq 1$, so the comparison test implies that $\sum n a_n x^{n-1}$ diverges. Thus the series have the same radius of convergence. $\square$

**Theorem 6.15.** Suppose that the power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n \qquad \text{for } |x-c| < R$$

has radius of convergence $R > 0$ and sum $f$. Then $f$ is differentiable in $|x - c| < R$ and

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1} \qquad \text{for } |x-c| < R.$$

**Proof.** The term-by-term differentiated power series converges in $|x - c| < R$ by Theorem 6.14. We denote its sum by

$$g(x) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1}.$$

Let $0 < \rho < R$. Then, by Theorem 6.2, the power series for $f$ and $g$ both converge uniformly in $|x-c| < \rho$. Applying Theorem 5.18 to their partial sums, we conclude that $f$ is differentiable in $|x - c| < \rho$ and $f' = g$. Since this holds for every $0 \leq \rho < R$, it follows that $f$ is differentiable in $|x-c| < R$ and $f' = g$, which proves the result. $\square$

Repeated application Theorem 6.15 implies that the sum of a power series is infinitely differentiable inside its interval of convergence and its derivatives are given by term-by-term differentiation of the power series. Furthermore, we can get an expression for the coefficients $a_n$ in terms of the function $f$; they are simply the Taylor coefficients of $f$ at $c$.

**Theorem 6.16.** If the power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

has radius of convergence $R > 0$, then $f$ is infinitely differentiable in $|x - c| < R$ and

$$a_n = \frac{f^{(n)}(c)}{n!}.$$

**Proof.** We assume $c = 0$ without loss of generality. Applying Theorem 6.16 to the power series

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \dots$$

$k$ times, we find that $f$ has derivatives of every order in $|x| < R$, and

$$\begin{aligned}
f'(x) &= a_1 + 2a_2 x + 3a_3 x^2 + \dots + n a_n x^{n-1} + \dots, \\
f''(x) &= 2a_2 + (3 \cdot 2) a_3 x + \dots + n(n-1) a_n x^{n-2} + \dots, \\
f'''(x) &= (3 \cdot 2 \cdot 1) a_3 + \dots + n(n-1)(n-2) a_n x^{n-3} + \dots, \\
&\vdots \\
f^{(k)}(x) &= (k!) a_k + \dots + \frac{n!}{(n-k)!} x^{n-k} + \dots,
\end{aligned}$$

where all of these power series have radius of convergence $R$. Setting $x = 0$ in these series, we get

$$a_0 = f(0), \quad a_1 = f'(0), \quad \dots \quad a_k = \frac{f^{(k)}(0)}{k!},$$

which proves the result (after replacing 0 by $c$). □

One consequence of this result is that convergent power series with different coefficients cannot converge to the same sum.

**Corollary 6.17.** If two power series

$$\sum_{n=0}^{\infty} a_n (x-c)^n, \qquad \sum_{n=0}^{\infty} b_n (x-c)^n$$

have nonzero-radius of convergence and are equal on some neighborhood of 0, then $a_n = b_n$ for every $n = 0, 1, 2, \dots$.

**Proof.** If the common sum in $|x - c| < \delta$ is $f(x)$, we have

$$a_n = \frac{f^{(n)}(c)}{n!}, \qquad b_n = \frac{f^{(n)}(c)}{n!},$$

since the derivatives of $f$ at $c$ are determined by the values of $f$ in an arbitrarily small open interval about $c$, so the coefficients are equal □

## 6.5. The exponential function

We showed in Example 6.8 that the power series

$$E(x) = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \dots + \frac{1}{n!} x^n + \dots.$$

has radius of convergence $\infty$. It therefore defines an infinitely differentiable function $E : \mathbb{R} \to \mathbb{R}$.

Term-by-term differentiation of the power series, which is justified by Theorem 6.15, implies that

$$E'(x) = 1 + x + \frac{1}{2!} x^2 + \dots + \frac{1}{(n-1)!} x^{(n-1)} + \dots,$$

so $E' = E$. Moreover $E(0) = 1$. As we show below, there is a unique function with these properties, and they are shared by the exponential function $e^x$. Thus, this power series provides an analytical definition of $e^x = E(x)$. All of the other familiar properties of the exponential follow from its power-series definition, and we will prove a few of them.

First, we show that $e^x e^y = e^{x+y}$. We continue to write the function as $E(x)$ to emphasise that we use nothing beyond its power series definition.

**Proposition 6.18.** For every $x, y \in \mathbb{R}$,

$$E(x)E(y) = E(x+y).$$

**Proof.** We have

$$E(x) = \sum_{j=0}^{\infty} \frac{x^j}{j!}, \qquad E(y) = \sum_{k=0}^{\infty} \frac{y^k}{k!}.$$

Multiplying these series term-by-term and rearranging the sum, which is justified by the absolute converge of the power series, we get

$$\begin{aligned} E(x)E(y) &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{x^j y^k}{j!\,k!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{x^{n-k} y^k}{(n-k)!\,k!}. \end{aligned}$$

From the binomial theorem,

$$\sum_{k=0}^{n} \frac{x^{n-k} y^k}{(n-k)!\,k!} = \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{(n-k)!\,k!} x^{n-k} y^k = \frac{1}{n!} (x+y)^n.$$

Hence,

$$E(x)E(y) = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} = E(x+y),$$

which proves the result. □

In particular, it follows that

$$E(-x) = \frac{1}{E(x)}.$$

Note that $E(x) > 0$ for all $x > 0$ since all the terms in its power series are positive, so $E(x) > 0$ for every $x \in \mathbb{R}$.

The following proposition, which we use below in Section 6.6.2, states that $e^x$ grows faster than any power of $x$ as $x \to \infty$.

**Proposition 6.19.** Suppose that $n$ is a non-negative integer. Then

$$\lim_{x \to \infty} \frac{x^n}{E(x)} = 0.$$

**Proof.** The terms in the power series of $E(x)$ are positive for $x > 0$, so for every $k \in \mathbb{N}$

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} > \frac{x^k}{k!} \qquad \text{for all } x > 0.$$

Taking $k = n+1$, we get for $x > 0$ that

$$0 < \frac{x^n}{E(x)} < \frac{x^n}{x^{(n+1)}/(n+1)!} = \frac{(n+1)!}{x}.$$

Since $1/x \to 0$ as $x \to \infty$, the result follows. ☐

Finally, we prove that the exponential is characterized by the properties $E' = E$ and $E(0) = 1$. This is a uniqueness result for an initial value problem for a simple linear ordinary differential equation.

**Proposition 6.20.** Suppose $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that

$$f' = f, \qquad f(0) = 1.$$

Then $f = E$.

**Proof.** Suppose that $f' = f$. Then using the equation $E' = E$, the fact that $E$ is nonzero on $\mathbb{R}$, and the quotient rule, we get

$$\left(\frac{f}{E}\right)' = \frac{fE' - Ef'}{E^2} = \frac{fE - Ef}{E^2} = 0.$$

It follows from Theorem 4.29 that $f/E$ is constant on $\mathbb{R}$. Since $f(0) = E(0) = 1$, we have $f/E = 1$, which implies that $f = E$. ☐

The logarithm can be defined as the inverse of the exponential. Other transcendental functions, such as the trigonometric functions, can be defined in terms of their power series, and these can be used to prove their usual properties. We will not do this in detail; we just want to emphasize that, once we have developed the theory of power series, we can define all of the functions arising in elementary calculus from the first principles of analysis.

## 6.6. Taylor's theorem and power series

Theorem 6.16 looks similar to Taylor's theorem, Theorem 4.41. There is, however, a fundamental difference. Taylor's theorem gives an expression for the error between a function and its Taylor polynomial of degree $n$. No questions of convergence are involved here. On the other hand, Theorem 6.16 asserts the convergence of an infinite power series to a function $f$, and gives an expression for the coefficients of the power series in terms of $f$. The coefficients of the Taylor polynomials and the power series are the same in both cases, but the Theorems are different.

Roughly speaking, Taylor's theorem describes the behavior of the Taylor polynomials $P_n(x)$ of $f$ as $x \to c$ with $n$ fixed, while the power series theorem describes the behavior of $P_n(x)$ as $n \to \infty$ with $x$ fixed.

**6.6.1. Smooth functions and analytic functions.** To explain the difference between Taylor's theorem and power series in more detail, we introduce an important distinction between smooth and analytic functions: smooth functions have continuous derivatives of all orders, while analytic functions are sums of power series.

**Definition 6.21.** Let $k \in \mathbb{N}$. A function $f : (a,b) \to \mathbb{R}$ is $C^k$ on $(a,b)$, written $f \in C^k(a,b)$, if it has continuous derivatives $f^{(j)} : (a,b) \to \mathbb{R}$ of orders $1 \le j \le k$. A function $f$ is smooth (or $C^\infty$, or infinitely differentiable) on $(a,b)$, written $f \in C^\infty(a,b)$, if it has continuous derivatives of all orders on $(a,b)$.

In fact, if $f$ has derivatives of all orders, then they are automatically continuous, since the differentiability of $f^{(k)}$ implies its continuity; on the other hand, the existence of $k$ derivatives of $f$ does not imply the continuity of $f^{(k)}$. The statement "$f$ is smooth" is sometimes used rather loosely to mean "$f$ has as many continuous derivatives as we want," but we will use it to mean that $f$ is $C^\infty$.

**Definition 6.22.** A function $f : (a,b) \to \mathbb{R}$ is analytic on $(a,b)$ if for every $c \in (a,b)$ $f$ is the sum in a neighborhood of $c$ of a power series centered at $c$ with nonzero radius of convergence.

Strictly speaking, this is the definition of a real analytic function, and analytic functions are complex functions that are sums of power series. Since we consider only real functions, we abbreviate "real analytic" to "analytic."

Theorem 6.16 implies that an analytic function is smooth: If $f$ is analytic on $(a,b)$ and $c \in (a,b)$, then there is an $R > 0$ and coefficients $(a_n)$ such that

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n \qquad \text{for } |x-c| < R.$$

Then Theorem 6.16 implies that $f$ has derivatives of all orders in $|x-c| < R$, and since $c \in (a,b)$ is arbitrary, $f$ has derivatives of all orders in $(a,b)$. Moreover, it follows that the coefficients $a_n$ in the power series expansion of $f$ at $c$ are given by Taylor's formula.

What is less obvious is that a smooth function need not be analytic. If $f$ is smooth, then we can define its Taylor coefficients $a_n = f^{(n)}(c)/n!$ at $c$ for every $n \ge 0$, and write down the corresponding Taylor series $\sum a_n(x-c)^n$. The problem is that the Taylor series may have zero radius of convergence, in which case it diverges for every $x \ne c$, or the power series may converge, but not to $f$.

### 6.6.2. A smooth, non-analytic function.

In this section, we give an example of a smooth function that is not the sum of its Taylor series.

It follows from Proposition 6.19 that if

$$p(x) = \sum_{k=0}^{n} a_k x^k$$

is any polynomial function, then

$$\lim_{x\to\infty} \frac{p(x)}{e^x} = \sum_{k=0}^{n} a_k \lim_{x\to\infty} \frac{x^k}{e^x} = 0.$$

We will use this limit to exhibit a non-zero function that approaches zero faster than every power of $x$ as $x \to 0$. As a result, all of its derivatives at 0 vanish, even though the function itself does not vanish in any neighborhood of 0. (See Figure 3.)

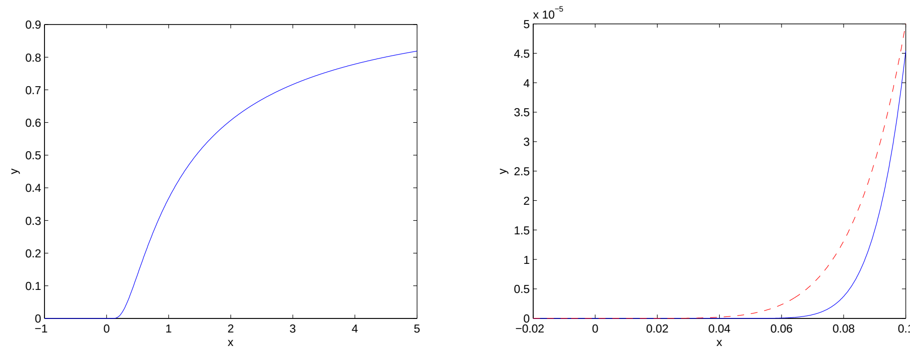


**Figure 3.** Left: Plot $y = \phi(x)$ of the smooth, non-analytic function defined in Proposition 6.23. Right: A detail of the function near $x = 0$. The dotted line is the power-function $y = x^6/50$. The graph of $\phi$ near 0 is "flatter' than the graph of the power-function, illustrating that $\phi(x)$ goes to zero faster than any power of $x$ as $x \to 0$.

**Proposition 6.23.** Define $\phi : \mathbb{R} \to \mathbb{R}$ by

$$\phi(x) = \begin{cases} \exp(-1/x) & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

Then $\phi$ has derivatives of all orders on $\mathbb{R}$ and

$$\phi^{(n)}(0) = 0 \qquad \text{for all } n \geq 0.$$

**Proof.** The infinite differentiability of $\phi(x)$ at $x \neq 0$ follows from the chain rule. Moreover, its $n$th derivative has the form

$$\phi^{(n)}(x) = \begin{cases} p_n(1/x)\exp(-1/x) & \text{if } x > 0, \\ 0 & \text{if } x < 0, \end{cases}$$

where $p_n(1/x)$ is a polynomial in $1/x$. (This follows, for example, by induction.) Thus, we just have to show that $\phi$ has derivatives of all orders at 0, and that these derivatives are equal to zero.

First, consider $\phi'(0)$. The left derivative $\phi'(0^-)$ of $\phi$ at 0 is clearly 0 since $\phi(0) = 0$ and $\phi(h) = 0$ for all $h < 0$. For the right derivative, writing $1/h = x$ and using Proposition 6.19, we get

$$\begin{aligned} \phi'(0^+) &= \lim_{h \to 0^+} \left[\frac{\phi(h) - \phi(0)}{h}\right] \\ &= \lim_{h \to 0^+} \frac{\exp(-1/h)}{h} \\ &= \lim_{x \to \infty} \frac{x}{e^x} \\ &= 0. \end{aligned}$$

Since both the left and right derivatives equal zero, we have $\phi'(0) = 0$.

To show that all the derivatives of $\phi$ at 0 exist and are zero, we use a proof by induction. Suppose that $\phi^{(n)}(0) = 0$, which we have verified for $n = 1$. The left derivative $\phi^{(n+1)}(0^-)$ is clearly zero, so we just need to prove that the right derivative is zero. Using the form of $\phi^{(n)}(h)$ for $h > 0$ and Proposition 6.19, we get that

$$\begin{aligned}
\phi^{(n+1)}(0^+) &= \lim_{h\to 0^+} \left[\frac{\phi^{(n)}(h) - \phi^{(n)}(0)}{h}\right] \\
&= \lim_{h\to 0^+} \frac{p_n(1/h)\exp(-1/h)}{h} \\
&= \lim_{x\to\infty} \frac{xp_n(x)}{e^x} \\
&= 0,
\end{aligned}$$

which proves the result. □

**Corollary 6.24.** The function $\phi : \mathbb{R} \to \mathbb{R}$ defined by

$$\phi(x) = \begin{cases} \exp(-1/x) & \text{if } x > 0, \\ 0 & \text{if } x \le 0, \end{cases}$$

is smooth but not analytic on $\mathbb{R}$.

**Proof.** From Proposition 6.23, the function $\phi$ is smooth, and the $n$th Taylor coefficient of $\phi$ at 0 is $a_n = 0$. The Taylor series of $\phi$ at 0 therefore converges to 0, so its sum is not equal to $\phi$ in any neighborhood of 0, meaning that $\phi$ is not analytic at 0. □

The fact that the Taylor polynomial of $\phi$ at 0 is zero for every degree $n \in \mathbb{N}$ does not contradict Taylor's theorem, which states that for $x > 0$ there exists $0 < \xi < x$ such that

$$\phi(x) = \frac{p_{n+1}(1/\xi)}{(n+1)!} e^{-1/\xi} x^{n+1}.$$

Since the derivatives of $\phi$ are bounded, this shows that for every $n \in \mathbb{N}$ there exists a constant $C_{n+1}$ such that

$$0 \le \phi(x) \le C_{n+1} x^{n+1} \qquad \text{for all } 0 \le x < \infty,$$

but this does not imply that $\phi(x) = 0$.

We can construct other smooth, non-analytic functions from $\phi$.

**Example 6.25.** The function

$$\psi(x) = \begin{cases} \exp(-1/x^2) & \text{if } x \ne 0, \\ 0 & \text{if } x = 0, \end{cases}$$

is infinitely differentiable on $\mathbb{R}$, since $\psi(x) = \phi(x^2)$ is a composition of smooth functions.

The following example is useful in many parts of analysis.

**Definition 6.26.** A function $f : \mathbb{R} \to \mathbb{R}$ has compact support if there exists $R \ge 0$ such that $f(x) = 0$ for all $x \in \mathbb{R}$ with $|x| \ge R$.

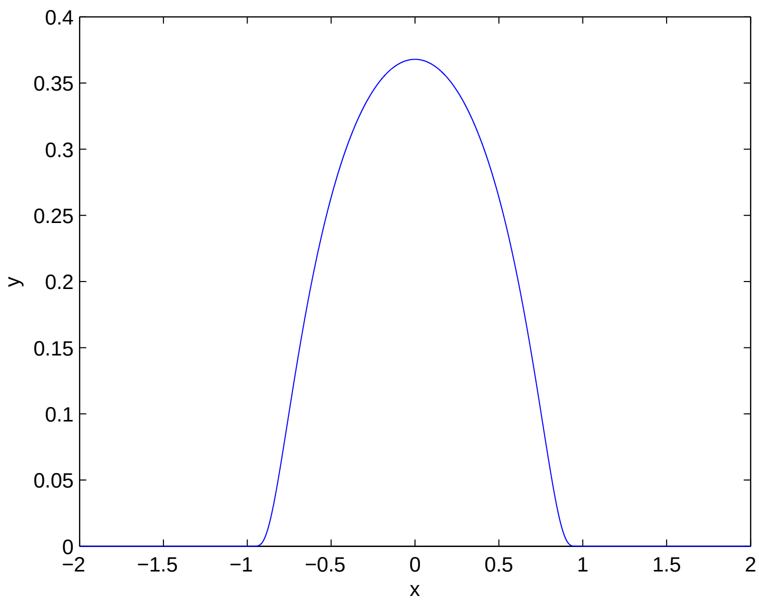


**Figure 4.** Plot of the smooth, compactly supported "bump" function defined in Example 6.27.

It isn't hard to construct continuous functions with compact support; one example that vanishes for $|x| \geq 1$ is

$$f(x) = \begin{cases} 1 - |x| & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

By matching left and right derivatives of a piecewise-polynomial function, we can similarly construct $C^1$ or $C^k$ functions with compact support. Using $\phi$, however, we can construct a smooth ($C^\infty$) function with compact support, which might seem unexpected at first sight.

**Example 6.27.** The function

$$\eta(x) = \begin{cases} \exp[-1/(1 - x^2)] & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

is infinitely differentiable on $\mathbb{R}$, since $\eta(x) = \phi(1 - x^2)$ is a composition of smooth functions. Moreover, it vanishes for $|x| \geq 1$, so it is a smooth function with compact support. Figure 4 shows its graph.

The function $\phi$ defined in Proposition 6.23 illustrates that knowing the values of a smooth function and all of its derivatives at one point does not tell us anything about the values of the function at other points. By contrast, an analytic function on an interval has the remarkable property that the value of the function and all of its derivatives at one point of the interval determine its values at all other points of the interval, since we can extend the function from point to point by summing its power series. (This claim requires a proof, which we omit.)

For example, it is impossible to construct an analytic function with compact support, since if an analytic function on $\mathbb{R}$ vanishes in any interval $(a, b) \subset \mathbb{R}$, then it must be identically zero on $\mathbb{R}$. Thus, the non-analyticity of the "bump"-function $\eta$ in Example 6.27 is essential.

## 6.7. Appendix: Review of series

We summarize the results and convergence tests that we use to study power series. Power series are closely related to geometric series, so most of the tests involve comparisons with a geometric series.

**Definition 6.28.** Let $(a_n)$ be a sequence of real numbers. The series

$$\sum_{n=1}^{\infty} a_n$$

converges to a sum $S \in \mathbb{R}$ if the sequence $(S_n)$ of partial sums

$$S_n = \sum_{k=1}^{n} a_k$$

converges to $S$. The series converges absolutely if

$$\sum_{n=1}^{\infty} |a_n|$$

converges.

The following Cauchy condition for series is an immediate consequence of the Cauchy condition for the sequence of partial sums.

**Theorem 6.29** (Cauchy condition)**.** The series

$$\sum_{n=1}^{\infty} a_n$$

converges if and only for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\left| \sum_{k=m+1}^{n} a_k \right| = |a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon \qquad \text{for all } n > m > N.$$

**Proof.** The series converges if and only if the sequence $(S_n)$ of partial sums is Cauchy, meaning that for every $\epsilon > 0$ there exists $N$ such that

$$|S_n - S_m| = \left| \sum_{k=m+1}^{n} a_k \right| < \epsilon \qquad \text{for all } n > m > N,$$

which proves the result. $\square$

Since

$$\left|\sum_{k=m+1}^{n} a_k\right| \leq \sum_{k=m+1}^{n} |a_k|$$

the series $\sum a_n$ is Cauchy if $\sum |a_n|$ is Cauchy, so an absolutely convergent series converges. We have the following necessary, but not sufficient, condition for convergence of a series.

**Theorem 6.30.** If the series

$$\sum_{n=1}^{\infty} a_n$$

converges, then

$$\lim_{n\to\infty} a_n = 0.$$

**Proof.** If the series converges, then it is Cauchy. Taking $m = n-1$ in the Cauchy condition in Theorem 6.29, we find that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|a_n| < \epsilon$ for all $n > N$, which proves that $a_n \to 0$ as $n \to \infty$. □

Next, we derive the comparison, ratio, and root tests, which provide explicit sufficient conditions for the convergence of a series.

**Theorem 6.31** (Comparison test)**.** Suppose that $|b_n| \leq a_n$ and $\sum a_n$ converges. Then $\sum b_n$ converges absolutely.

**Proof.** Since $\sum a_n$ converges it satisfies the Cauchy condition, and since

$$\sum_{k=m+1}^{n} |b_k| \leq \sum_{k=m+1}^{n} a_k$$

the series $\sum |b_n|$ also satisfies the Cauchy condition. Therefore $\sum b_n$ converges absolutely. □

**Theorem 6.32** (Ratio test)**.** Suppose that $(a_n)$ is a sequence of real numbers such that $a_n$ is nonzero for all sufficiently large $n \in \mathbb{N}$ and the limit

$$r = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

exists or diverges to infinity. Then the series

$$\sum_{n=1}^{\infty} a_n$$

converges absolutely if $0 \leq r < 1$ and diverges if $1 < r \leq \infty$.

**Proof.** If $r < 1$, choose $s$ such that $r < s < 1$. Then there exists $N \in \mathbb{N}$ such that

$$\left|\frac{a_{n+1}}{a_n}\right| < s \qquad \text{for all } n > N.$$

It follows that

$$|a_n| \leq M s^n \qquad \text{for all } n > N$$

where $M$ is a suitable constant. Therefore $\sum a_n$ converges absolutely by comparison with the convergent geometric series $\sum M s^n$.

If $r > 1$, choose $s$ such that $r > s > 1$. There exists $N \in \mathbb{N}$ such that

$$\left|\frac{a_{n+1}}{a_n}\right| > s \qquad \text{for all } n > N,$$

so that $|a_n| \geq Ms^n$ for all $n > N$ and some $M > 0$. It follows that $(a_n)$ does not approach 0 as $n \to \infty$, so the series diverges. □

Before stating the root test, we define the lim sup.

**Definition 6.33.** If $(a_n)$ is a sequence of real numbers, then

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} b_n, \qquad b_n = \sup_{k\geq n} a_k.$$

If $(a_n)$ is a bounded sequence, then $\limsup a_n \in \mathbb{R}$ always exists since $(b_n)$ is a monotone decreasing sequence of real numbers that is bounded from below. If $(a_n)$ isn't bounded from above, then $b_n = \infty$ for every $n \in \mathbb{N}$ (meaning that $\{a_k : k \geq n\}$ isn't bounded from above) and we write $\limsup a_n = \infty$. If $(a_n)$ is bounded from above but $(b_n)$ diverges to $-\infty$, then $(a_n)$ diverges to $-\infty$ and we write $\limsup a_n = -\infty$. With these conventions, every sequence of real numbers has a lim sup, even if it doesn't have a limit or diverge to $\pm\infty$.

We have the following equivalent characterization of the lim sup, which is what we often use in practice. If the lim sup is finite, it states that every number bigger than the lim sup eventually bounds all the terms in a tail of the sequence from above, while infinitely many terms in the sequence are greater than every number less than the lim sup.

**Proposition 6.34.** Let $(a_n)$ be a real sequence with

$$L = \limsup_{n\to\infty} a_n.$$

(1) If $L \in \mathbb{R}$ is finite, then for every $M > L$ there exists $N \in \mathbb{N}$ such that $a_n < M$ for all $n > N$, and for every $m < L$ there exist infinitely many $n \in \mathbb{N}$ such that $a_n > m$.

(2) If $L = -\infty$, then for every $M \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that $a_n < M$ for all $n > N$.

(3) If $L = \infty$, then for every $m \in \mathbb{R}$, there exist infinitely many $n \in \mathbb{N}$ such that $a_n > m$.

**Theorem 6.35** (Root test)**.** Suppose that $(a_n)$ is a sequence of real numbers and let

$$r = \limsup_{n\to\infty} |a_n|^{1/n}.$$

Then the series

$$\sum_{n=1}^{\infty} a_n$$

converges absolutely if $0 \leq r < 1$ and diverges if $1 < r \leq \infty$.

**Proof.** First suppose $0 \leq r < 1$. If $0 < r < 1$, choose $s$ such that $r < s < 1$, and let

$$t = \frac{r}{s}, \qquad r < t < 1.$$

If $r = 0$, choose any $0 < t < 1$. Since $t > \limsup |a_n|^{1/n}$, Proposition 6.34 implies that there exists $N \in \mathbb{N}$ such that

$$|a_n|^{1/n} < t \qquad \text{for all } n > N.$$

Therefore $|a_n| < t^n$ for all $n > N$, where $t < 1$, so it follows that the series converges by comparison with the convergent geometric series $\sum t^n$.

Next suppose $1 < r \leq \infty$. If $1 < r < \infty$, choose $s$ such that $1 < s < r$, and let

$$t = \frac{r}{s}, \qquad 1 < t < r.$$

If $r = \infty$, choose any $1 < t < \infty$. Since $t < \limsup |a_n|^{1/n}$, Proposition 6.34 implies that

$$|a_n|^{1/n} > t \qquad \text{for infinitely many } n \in \mathbb{N}.$$

Therefore $|a_n| > t^n$ for infinitely many $n \in \mathbb{N}$, where $t > 1$, so $(a_n)$ does not approach zero as $n \to \infty$, and the series diverges. $\square$