Integrals Depending on a Parameter

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1 Proper Integrals Depending on a Parameter

1.1 The Concept of an Integral Depending on a Parameter

An integral depending on a parameter is a function of the form

$$F(t) = \int_{E_t} f(x,t)\,\mathrm{d}t, \tag{1}$$

where t plays the role of a parameter ranging over a set T, and to each value $t \in T$ there corresponding a set E_t and a function $\varphi_t(x) = f(x,t)$ that is integrable over E_t in the proper or improper sense. The nature of the set T may be quite varied, but of course the most important cases occurs when T is a subset of $\mathbb{R}, \mathbb{C}, \mathbb{R}^n, \mathbb{C}^n$.

If the integral (Eq. 1) is a proper integral for each value of the parameter $t \in T$, we say that the function in Eq. 1 is a proper integral depending on a parameter.

But if the integral in Eq. 1 exists only as an improper integral for some or all of the value of $t \in T$, we usually call F an improper integral depending on a parameter.

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1.2 Continuity of an Integral Depending on a Parameter

Proposition 1. *Let $P = \{(x, y) \in \mathbb{R}^2 | a \leq x \leq b, c \leq y \leq d\}$ be a rectangle on the plane $\mathbb{R}^2$. If the function $f : P \to \mathbb{R}$ is continuous, that is, if $f \in C(P, \mathbb{R})$, then the function*

$$F(y) = \int_a^b f(x, y) \, \mathrm{d}x \tag{2}$$

is continuous at every point $y \in [c, d]$.

Example 1. *Find the limit*

$$\lim_{a \to 0} \int_0^1 \frac{dx}{1 + x^2 \cos ax}$$

Proposition 2. *Suppose $f(x, y) \in C([a, b] \times [c, d])$, then*

$$\int_c^d \mathrm{d}y \int_a^b f(x, y) \, \mathrm{d}x = \int_a^b \mathrm{d}x \int_c^d f(x, y) \, \mathrm{d}y.$$

Example 2. *Find the value*

$$I = \int_0^1 \frac{x^b - x^a}{\ln x} \, \mathrm{d}x, for \quad b > a > 0$$

1.3 Differentiation of an Integral Depending on a Parameter

Proposition 3. *If the function $f : P \to \mathbb{R}$ is continuous and has a continuous partial derivative with respect to y on the rectangle $P = \{(x, y) \in \mathbb{R}^2 | a \leq x \leq b, c \leq y \leq d\}$, then the integral of Eq. 2 belongs to $C^{(1)}([c, d], \mathbb{R})$, and*

$$F'(y) = \int_a^b \frac{\partial f}{\partial y}(x, y) \, \mathrm{d}x.$$

Example 3. *The complete elliptic integrals*

$$E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \varphi} \, \mathrm{d}\varphi, K(k) = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}$$

as functions of the parameter $k, 0 < k < 1$, called the modulus of the corresponding *elliptic integral, are connected by the relations*

$$\frac{\mathrm{d}E}{\mathrm{d}k} = \frac{E-K}{k}, \frac{\mathrm{d}K}{\mathrm{d}k} = \frac{E}{k(1-k^2)} - \frac{K}{k}.$$

Proposition 4. *Suppose the function $f : P \to \mathbb{R}$ is continuous and has a continuous partial derivative $\frac{\partial f}{\partial y}$ on the rectangle $P = \{(x, y) \in \mathbb{R}^2 | a \leq x \leq b, c \leq y \leq d\}$, further suppose $\alpha(y), \beta(y)$ are continuously differentiable functions on $[c, d]$ whose values lie in $[a, b]$ for every $y \in [c, d]$. Then the integral*

$$F(y) = \int_{\alpha(y)}^{\beta(y)} f(x, y)\,\mathrm{d}x$$

is defined for every $y \in [c, d]$ and belongs to $C^{(1)}([c, d])$, and the following formula holds;

$$F'(y) = f(\beta(y), y) \cdot \beta'(y) - f(\alpha(y), y) \cdot \alpha'(y) + \int_{\alpha(y)}^{\beta(y)} \frac{\partial f}{\partial y}(x, y)\,\mathrm{d}x.$$

Example 4. *Let*

$$F_n(x) = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} f(t)\,\mathrm{d}t,$$

where $n \in N$ and f is a function that is continuous on the interval of integration. Let us verify that $F_n^n(x) = f(x)$.

1.4 Integration of an Integral Depending on a Parameter

Proposition 5. *If the function $f : P \to \mathbb{R}$ is continuous in the rectangle $P = \{(x, y) \in \mathbb{R}^2 | a \leq x \leq bandc \leq y \leq d\}$, then the integral Eq. 2 is integrable over the closed interval $[c, d]$ and the following equality holds:*

$$\int_c^d \int_a^b f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_a^b \int_c^d f(x, y)\,\mathrm{d}y\,\mathrm{d}x \tag{3}$$

2 Improper Integrals Depending on a Parameter

2.1 Uniform Convergence of an Improper Integral With Respect to a Parameter

a. Basic Definition and Examples Suppose that the improper integral

$$F(y) = \int_a^{\omega} f(x, y)\,\mathrm{d}x \tag{4}$$

over the interval $[a, \omega]$ converges for each value $y \in Y$. For definiteness we shall assume that the integral Eq. 4 has only one singularity and that it involves the upper limit of the integration (that is, either $\omega = +\infty$ or the function f is unbounded as a function of x in a neighbourhood of ω.)

Definition 1. *We say that the improper integral Eq. 4 depending on the parameter $y \in Y$ converges uniformly on the set $E \subset Y$ if for every $\epsilon > 0$ there exists a neighborhood $U_{[a,\omega[}(\omega)$ of ω in the set $[a, \omega[$ such that the estimate*

$$\left| \int_b^{\omega} f(x, y)\,\mathrm{d}x \right| < \epsilon \tag{5}$$

for the remainder of the integral Eq.4 holds for every $b \in U_{[a,\omega[}(\omega)$ and every $y \in E$.

If we introduce the notation

$$F_b(y) = \int_a^{b} f(x, y)\,\mathrm{d}x \tag{6}$$

for a proper integral approximating the improper integral of Eq.4, the basic definition of this section can be restated as in a different form equivalent to the previous one: uniform convergence of the integral of Eq. 4 on the set $E \subset Y$ by definition means that

$$F_b(y) \rightrightarrows F(y) \text{ on } E \text{ as } b \to \omega, b \in [a, \omega[\tag{7}$$

Example 5. *The integral*

$$\int_1^{+\infty} \frac{\mathrm{d}x}{x^2 + y^2}$$

converges uniformly on the entire set $\mathbb{R}$ of values of the parameter $y \in \mathbb{R}$.

Example 6. *The integral*
$$\int_0^{+\infty} e^{-xy}\,\mathrm{d}x$$
converges only when $y > 0$. *Moreover it converges uniformly on every set* $\{y \in \mathbb{R} | y \geq y_0 \geq 0\}$.

Example 7. *Let us show that each of the integrals*

$$\Phi(x) = \int_0^{+\infty} x^\alpha y^{\alpha+\beta+1} e^{-(1+x)y}\,\mathrm{d}y$$

$$F(y) = \int_0^{+\infty} x^\alpha y^{\alpha+\beta+1} e^{-(1+x)y}\,\mathrm{d}x$$

in which α, β *are fixed positive numbers, converges uniformly on the set of non-negative values of the parameter.*

b. The Cauchy Criterion for Uniform Convergence of an Integral

Proposition 6. ***Cauchy Criterion.*** *A necessary and sufficient condition for the improper integral of Eq. 4 depending on parameter* $y \in Y$ *to converge uniformly on a set* $E \subset Y$ *is that for every* $\epsilon > 0$ *there exist a neighborhood* $U_{[a,\omega[}$ *of the point* ω *such that*
$$\left| \int_{b_1}^{b_2} f(x,y) \right| < \epsilon$$
for every $b_1, b_2 \in U_{[a,\omega[}$ *and every* $y \in E$.

Corollary 1. *If the function* f *in the integral of Eq. 4 is continuous on the set* $[a,\omega[\times[c,d]$ *and the integral of Eq. 4 converges for every* $y \in]c,d[$ *but diverges for* $y = c$ *or* $y = d$, *then it converges non-uniformly on the interval* $]c,d[$ *and also on any set* $E \subset]c,d[$ *whose closure contains the point of divergence.*

Example 8. *The integral*
$$\int_0^{+\infty} e^{-tx^2}\,\mathrm{d}x$$
converges for $t > 0$ *and diverges at* $t = 0$, *hence it demonstrably converges non-uniformly on every set of positive numbers having* 0 *as a limit point.*

c. Sufficient Conditions for Uniform Convergence of an Improper Integral Depending on a Parameter

Proposition 7. *The Weierstrass test. Suppose the functions $f(x,y)$ and $g(x,y)$ are integrable with respect to x on every closed interval $[a,b] \subset [a,\omega[$ for each value of $y \in Y$.*

If the inequality $|f(x,y)| \leq g(x,y)$ holds for each value of $y \in Y$ and every $x \in [a,\omega[$ and the integral

$$\int_a^\omega g(x,y)\,\mathrm{d}x$$

converges uniformly on Y, then the integral

$$\int_0^\omega f(x,y)\,\mathrm{d}x$$

converges absolutely for each $y \in Y$ and uniformly on Y.

The most frequently encountered case of Proposition 2 occurs when the function g is independent of the parameter y. It is this case in which Proposition 2 is usually called the Weierstrass M-test for uniform convergence of an integral.

Example 9. *The integral*

$$\int_0^\infty \frac{\cos\alpha x}{1+x^2}\,\mathrm{d}x$$

converges uniformly on the whole set $\mathbb{R}$ of the parameter α, since $\left|\frac{\cos\alpha x}{1+x^2}\right| \leq \frac{1}{1+x^2}$, and the integral $\int_0^\infty \frac{\mathrm{d}x}{1+x^2}$ converges.

Proposition 8. *(*Abel-Dirichlet test.*) Assume that the function $f(x,y)$ and $g(x,y)$ are integrable with respect to x at each $y \in Y$ on every closed interval $[a,b] \subset [a,\omega[$.*

A sufficient condition for uniform convergence of the integral

$$\int_a^\omega (f \cdot g)\,\mathrm{d}x$$

on the set Y is that one of the following two pairs of conditions holds:

1-1) *either there exists a constant $M \in \mathbb{R}$ such that*

$$\left|\int_a^b f(x,y)\,\mathrm{d}x\right| < M$$

for any $b \in [a,\omega[$ and any $y \in Y$ and

1-2) *for each* $y \in Y$ *the function* $g(x,y)$ *is monotonic with respect to* x *on the interval* $[a,\omega[$ *and* $g(x,y) \rightrightarrows 0$ *on* Y *as* $x \to \omega$, $x \in [a,\omega[$, *or*

2-1) *the integral*

$$\int_a^\omega f(x,y)\,\mathrm{d}x$$

converges uniformly on the set Y *and*

2-2) *for each* $y \in Y$ *the function* $g(x,y)$ *is monotonic with respect to* x *on the interval* $[a,\omega[$ *and there exists a constant* $M \in \mathbb{R}$ *such that*

$$|g(x,y)| < M$$

for every $x \in [a,\omega[$ *and every* $y \in Y$.

Applying the second mean-value theorem for the integral, we have

$$\int_{b_1}^{b_2} (f \cdot g)(x,y)\,\mathrm{d}x = g(b_1,y)\int_{b_1}^{\xi} f(x,y)\,\mathrm{d}x + g(b_2,y)\int_{\xi}^{b_2} f(x,y)\,\mathrm{d}x$$

Example 10. *The integral*

$$\int_0^\infty \frac{\sin x}{x} e^{-xy}\,\mathrm{d}x$$

converges uniformly on the set $\{y \in \mathbb{R} | y \geq 0\}$.

Example 11. *The integral*

$$\int_0^\infty \frac{\sin xy}{x}\,\mathrm{d}x$$

converges uniformly on the set $\{y \in \mathbb{R} | y \geq y_0 > 0\}$ *and not uniformly convergence on the set* $\{y \in \mathbb{R} | y > 0\}$

Example 12. The integrali $\displaystyle\int_0^{+\infty} \frac{\cos x^2}{x^p}\,\mathrm{d}x$ converges uniformly on each $p \in [\alpha,\beta] \subset (-1,1)$.

3 Limiting Passage under the Sign of an Improper Integral and Continuity of an Improper Integral Depending on a Parameter

Proposition 9. *Let $f(x,y)$ be a family of functions depending on a parameter $y \in Y$ that are integrable, possibly in the improper sense, on the interval $a \le x \le \omega$, and let $\mathcal{B}_Y$ be a base in Y.*

If

a) for every $b \in [a, \omega[$

$$f(x,y) \rightrightarrows \varphi(x) \text{ on } [a,b] \text{ over the base } \mathcal{B}_Y,$$

b) the integral $\displaystyle\int_a^\omega f(x,y)\,\mathrm{d}x$ converges uniformly on Y, then the limit function $\varphi$is improperly integrable on $[a,\omega[$ and the following equality holds:

$$\lim_{\mathcal{B}_Y} \int_a^\omega f(x,y)\,\mathrm{d}x = \int_a^\omega \varphi\,\mathrm{d}x.$$

Proposition 10. *If*

a) the function $f(x,y)$ is continuous on the set

$\{(x,y) \in \mathbb{R}^2 | a \le x < \omega, c \le y \le d\}$ *and*

b) the integral $F(y) = \int_a^\omega f(x,y)\,\mathrm{d}x$ converges uniformly on $[c,d]$,

then the function $F(y)$ is continuous on $[c,d]$.

Proposition 11. *Suppose $f(x,y)$ is continuous on $[a,+\infty) \times [c,d]$, and the integral $\int_a^\infty f(x,y)\,\mathrm{d}x$ converges uniformly on $[c,d]$, then we have*

$$\int_c^d \mathrm{d}y \int_a^{+\infty} f(x,y)\,\mathrm{d}x = \int_a^{+\infty} \mathrm{d}x \int_c^d f(x,y)\,\mathrm{d}y$$

Proposition 12. *Suppose $f(x,y), f_y(x,y)$ are continuous on $[a,+\infty) \times [c,d]$, for each $y \in [c,d]$ the integral $\int_a^{+\infty} f(x,y)\,\mathrm{d}x$ converges. Furthermore the integral*

$\int_a^{+\infty} f_y(x,y)\,\mathrm{d}x$ *is uniformly converges. Then we have*

$$\frac{d}{dy}\int_a^{+\infty} f(x,y)\,\mathrm{d}x = \int_a^{+\infty} \frac{\partial}{\partial y} f(x,y)\,\mathrm{d}x$$

4 The Eulerian Integrals

In this section and the next we shall illustrate the application of the theory developed above to some specific integrals of importance in analysis that depend on a parameter.

Following Legendre, we define the Eulerian integrals of first and second kinds respectively as the two special functions that follow:

$$B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x \tag{8}$$

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1}e^{-x}\,\mathrm{d}x. \tag{9}$$

The first of these is called the beta function and the second the gamma function.

4.1 The Beta Function

a. Domain of Definition A necessary and sufficient condition for the convergence of the integral of the beta function at the lower limit is that $\alpha > 0$. Similarly, convergence at 1 occurs if and only if $\beta > 0$. Thus the beta function is defined when both of the following conditions hold simultaneously:

$$\alpha > 0 \text{ and } \beta > 0$$

b. Symmetry We can verify that:

$$B(\alpha,\beta) = B(\beta,\alpha)$$

c. The Reduction Formula If $\alpha > 1$, the following equality holds:

$$B(\alpha,\beta) = \frac{\alpha-1}{\alpha+\beta-1}B(\alpha-1,\beta)$$

We can now write the reduction form:

$$B(\alpha,\beta) = \frac{\alpha-1}{\alpha+\beta-1}B(\alpha,\beta-1)$$

It can be seen immediately from the definition of the beta function that $B(\alpha,1) = \frac{1}{\alpha}$, and so for $n \in \mathbb{N}$ we obtain

$$\begin{aligned} B(\alpha,n) &= \frac{n-1}{\alpha+n-1}\cdot\frac{n-2}{\alpha+n-2}\cdots\frac{n-(n-1)}{\alpha+n-(n-1)}B(\alpha,1) \\ &= \frac{(n-1)!}{\alpha(\alpha+1)\cdots(\alpha+n-1)}. \end{aligned} \tag{10}$$

In particular, for $m, n \in \mathbb{N}$

$$B(m,n) = \frac{(m-1)!(n-1)!}{(m+n-1)!} \tag{11}$$

d. Other forms of Representation of the Beta Function (1) One form for the beta function is

$$B(\alpha,\beta) = 2\int_0^{\frac{\pi}{2}} \cos^{2p-1}\phi \sin^{2q-1}\phi \,\mathrm{d}\phi$$

$$B\left(\frac{1}{2},\frac{1}{2}\right) = \pi$$

(2) The other form for the beta function is

$$B(\alpha,\beta) = \int_0^{+\infty} \frac{t^{\beta-1}}{(1+t)^{\alpha+\beta}}\,\mathrm{d}t$$

5 The Gamma Function

a. Domain of the Definition The Gamma Function is:

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1} e^{-x} \, \mathrm{d}x$$

It can be seen from the definition that the integral defining the gamma function converges at zero only for $\alpha > 0$, while it converges at infinity for all values of $\alpha \in \mathbb{R}$, due to the presence of the rapidly decreasing factor e^{-x}. Thus the gamma function is defined for $\alpha > 0$.

b. Smoothness and the Formula for the Derivatives The gamma function is infinitely differentiable, and

$$\Gamma^{(n)}(\alpha) = \int_0^{+\infty} x^{\alpha-1} \ln^n x e^{-x} \, \mathrm{d}x \tag{12}$$

c. The Reduction Formula The relation

$$\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$$

holds. It is known as the reduction formula for the gamma function.

Since $\Gamma(1) = 1$, we conclude that for $n \in \mathbb{N}$

$$\Gamma(n + 1) = n!$$

Thus the gamma function turns out to be closely connected with the number-theoretic function $n!$.

d. The Euler-Gauss Formula This is usually given to the following equality:

$$\Gamma(\alpha) = \lim_{n\to\infty} n^\alpha \frac{(n-1)!}{\alpha(\alpha+1)\cdots(\alpha+n-1)} \tag{13}$$

e. The Complement Formula For $0 < \alpha < 1$ the values α and $1 - \alpha$ of the argument of the gamma function are mutually complementary, so that the equality

$$\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin \pi\alpha}(0 < \alpha < 1) \tag{14}$$

It follows in particular from EQ 14 that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

We observe that

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{+\infty} x^{-\frac{1}{2}} e^{-x} \, \mathrm{d}x = 2\int_0^{+\infty} e^{-u^2} \, \mathrm{d}u = \sqrt{\pi}$$

f. Connection Between the Beta and Gamma Function The connection between the beta and gamma function is

$$B\left(\alpha, \beta\right) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{15}$$

Example 13. *Find the result of*

$$I = \int_0^{\frac{\pi}{2}} \sin^6 x \cos^4 x \, \mathrm{d}x$$

Example 14. *Find the result of*

$$\int_0^1 x^8 \sqrt{1-x^3} \, \mathrm{d}x$$

Example 15. *suppose $\alpha > -1$, find the results of the following integrals*

$$\int_0^{\frac{\pi}{2}} \sin^\alpha x \, \mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos^\alpha x \, \mathrm{d}x$$

Furthermore, find the volume of the n-dimensional sphere of the form

$$B_n = \left\{(x_1, x_2, \cdots, x_n) | x_1^2 + x_2^2 + \cdots + x_n^2 \leq R^2\right\}$$